CONVOLUTION :

Defn.

If $f(t)$ and $g(t)$ are two functions defined for $t \geq 0$ then the convolution of $f(t)$ & $g(t)$ is defined as $f(t) * g(t) = (f * g)(t) = \int_0^t f(u)\, g(t-u)\, du$

NOTE: $f(t) * g(t) = g(t) * f(t)$

CONVOLUTION THEOREM :

If $f(t)$ & $g(t)$ are two laplace transformable functions defined for $t \geq 0$ then

$L[f(t) * g(t)]$ is given by

$$L[f(t) * g(t)] = L[f(t)] * L[g(t)]$$

$$L^{-1}[F(s) \cdot G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$$

(1) Using convolution theorem find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

Soln:

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right]\cdot L^{-1}\left[\frac{1}{s^2+a^2}\right] \quad \text{[By Convoln. thm]}$$

$$= \cos at \cdot \frac{\sin at}{a}$$

$$= \frac{1}{a}\cdot \cos at \cdot \sin at$$

$$= \frac{1}{a}\int_0^t \cos au \sin a(t-u)\, du \quad \text{[By Convoln. defn]}$$

$$= \frac{1}{2a}\int_0^t [\sin at - \sin(2au - at)]\, du$$

$$= \frac{1}{2a}\left\{u\sin at + \frac{\cos(2au-at)}{2a}\right\}_0^t$$

$$= \frac{1}{2a}\left[t\sin at + \frac{\cos at}{2a} - 0 - \frac{\cos at}{2a}\right]$$

$$= \frac{1}{2a}\, t\sin at.$$

Find the inverse LT by convolution thm. $\frac{1}{s^2(s+5)}$

$$L^{-1}\left[\frac{1}{s^2(s+5)}\right] = L^{-1}\left[\frac{1}{s^2}\right] * L^{-1}\left[\frac{1}{s+5}\right]$$

$$= t * e^{-5t}$$

$$= \int_0^t u e^{-5(t-u)} du$$

$$= e^{-5t} \int_0^t u e^{5u} du$$

$$= e^{-5t}\left[u\frac{e^{5u}}{5} - \frac{e^{5u}}{25}\right]_0^t$$

$$= e^{-5t}\left[t\frac{e^{5t}}{5} - \frac{e^{5t}}{25} - \left(-\frac{1}{25}\right)\right]$$

$$= \frac{t}{5} - \frac{1}{25} + \frac{e^{-5t}}{25}$$

$$= \frac{1}{25}\left[5t + e^{-5t} - 1\right]$$

3) using convolution theorem find

i) $\frac{1}{(s^2+a^2)^2}$ soln: $\frac{1}{2a^2}\left[\frac{\sin at}{a} - t\cos at\right]$

ii) $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ soln: $\frac{a\sin at - b\sin bt}{a^2-b^2}$