## Laurent's Series

7 (or) 13M

Let $c_1$ and $c_2$ be two eccentric circles $|z-a| = R_1$, $|z-a| = R_2$ where $R_2 < R_1$. let $f(z)$ be analytic on $c_1$ and $c_2$ in the angular region R between them. Then for point z in R,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

$$a_n = \frac{1}{2\pi i} \int_{c_1} \frac{f(z)}{(z-a)^{1+n}} dz$$

$$b_n = \frac{1}{2\pi i} \int_{c_2} \frac{f(z)}{(z-a)^{-n}} dz$$ where, the integral being taken in anticlock wise direction.

**note :-**

In laurent's series of $f(z)$, with the terms containing positive power which is regular part and the terms containing which negative power is called principle part.

1) Expand $f(z) = \frac{7z-2}{z(z-2)(z+1)}$ in laurent's series

If i) $|z| < 2$ ii) $|z| > 3$ iii) $2 < |z| < 3$

iv) $1 < |z+1| < 3$

soln:- By partial fraction method

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1} \rightarrow (1)$$

$$= \frac{A(z-2)(z+1) + Bz(z+1) + Cz(z-2)}{z(z-2)(z+1)}$$

$$7z-2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2) \rightarrow (2)$$

Put $z=2$,

$$14-2 = 0 + 2B(3) + 0$$
$$12 = 6B$$
$$B = 2$$

Put $z=-1$

$$-7-2 = A(0) + B(0) + C(-1)(-1-2)$$
$$-9 = 3C$$
$$C = -3$$

Put $z=0$,

$$0-2 = A(-2)(1) + 0 + 0$$
$$-2 = -2A$$
$$A = 1$$

$$(1) \Rightarrow \frac{7z-2}{z(z-2)(z+1)} = \frac{1}{z} + \frac{2}{z-2} + \frac{-3}{z+1}$$

$$(1) \Rightarrow \frac{7z-2}{z(z-2)(z+1)} = 1/z + 2/z-2 - 3/z+1$$

i) $\frac{|z|}{2} < 1$

$$f(z) = \frac{7z-2}{z(z-2)(z+1)} = 1/z + \frac{2}{-2(z/_{-2}+1)} - \frac{3}{(1+z)}$$

$$= 1/z - (1-z/_{3})^{-1} - 3(1+z)^{-1}$$

$= 1/z - \left(1 + z/2 + \left(\frac{z}{2}\right)^2 + \dots\right)$
$- 3\left(1 - z + z^2 - \dots\right)$

$f(z) = 1/z - \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n - 3 \sum_{n=0}^{\infty} (-1)^n z^n$

ii) $|z| > 2$

$2 < |z| \Rightarrow \frac{3}{|z|} < 1$

$f(z) = 1/z + 2/z-2 - 3/z+1$

$= 1/z + \frac{2}{z(1-2/z)} - \frac{3}{z(1+1/z)}$

$= 1/z + 2/z\ (1 - 2/z)^{-1} - 3/z\ (1 + 1/z)^{-1}$

$= 1/z + 2/z\ \left(1 + 2/z + \left(2/z\right)^2 + \dots\right) - 3/z$
$\left(1 - 1/z + (1/z)^2 \dots\right)$

$f(z) = 1/z = 2/z \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n - \frac{3}{2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{z}\right)^n$

iii) $2 < |z| < 3$

$2 < |z|$ and $|z| < 3$

$\frac{2}{|z|} < 1 \qquad \frac{|z|}{3} < 1$

$f(z) = 1/z + \frac{2}{z-2} - \frac{3}{z+1}$

$f(z) = \frac{1}{z} + \frac{2}{z\left(1 - \frac{2}{z}\right)} - \frac{3}{z\left(1 + \frac{1}{z}\right)}$

$f(z) = 1/z + 2/z\ (1 - 2/z)^{-1} - 3/z\ (1 + 1/z)^{-1}$

$= 1/z + 2/z\ \left(1 + 2/z + (2/z)^2 + \dots\right) - 3/2 \left(1 - 1/z + \left(\frac{1}{z}\right)^2 + \dots\right)$

$f(z) = 1/z + 2/z \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n - 3/z \sum_{n=0}^{\infty} (-1)^n (1/2)^n$

iv) $1 < |z+1| < 3$

Take $z+1 = u$

$z = \frac{u-1}{1}$

$f(z) = \frac{1}{u-1} + \frac{2}{u-1-2} - \frac{3}{u-1+1}$

$f(z) = \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u}$

$1 < |u|$ and $|u| < 3$

$1/|u| < 1$ and $\frac{|u|}{3} < 1$

$f(z) = \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u}$

$f(z) = 1/u-1 + 2/u-3 - 3/u$

$f(z) = \frac{1}{u(1-1/u)} + \frac{2}{-3(\frac{u}{-3}+1)} - \frac{3}{u}$

$f(z) = \frac{1}{u}(1-1/u)^{-1} - 2/3\,(1-u/3)^{-1} - 3/u$

$f(z) = 1/u\,(1+1/u+(1/u)^2+\dots) - 2/3\,(1+u/3+(u/3)^2+\dots - 3/u$

$f(z) = \frac{1}{z+1}\left(1+\frac{1}{(z+1)}+\frac{1}{(z+1)^2}+\dots\right) - 2/3\left(1+\frac{(z+1)}{3}+\left(\frac{(z+1)}{3}\right)^2+\dots\right) - 3/(z+1)$

$f(z) = \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n = \frac{2}{z}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n \frac{3}{(z+1)}$

2) Expand $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ in the laurent's series interval

i) $|z| > 3$, ii) $2 < |z| < 3$

By partial function method :-

$f(z) = \dfrac{z^2-1}{(z+2)(z+3)} = A + \dfrac{B}{z+2} + \dfrac{C}{z+3}$ ( If the deg is

since in both then add a constant)

$= \dfrac{A(z+2)(z+3)+B(z+3)+C(z+2)}{(z+2)(z+3)}$

$z^2-1 = A(z+2)(z+3) + B(z+3) + C(z+2) \longrightarrow (2)$

put $z=-2$, $\qquad z+2=0$, $z=-2$

$(-2)^2-1 = 0 + B(-2+3) + 0 \qquad z+3=0$, $z=-3$

$3 = B \Rightarrow B=3$

Put $z=-3$,

$(-3)^2-1 = 0+0+C(-3+2)$

$8 = -C \Rightarrow \boxed{C=-8}$

iii) $z=0$,

$(0)^2-1 = A(0+2)(0+3) + B(0+3) + C(0+2)$

$-1 = 6A + 3B + 2C$

$-1 = 6A + 3(3) + 2(-8)$

$-1 = 6A + 9 - 16$

$-1 = 6A - 7$

$-1+7 = 6A$

$6 = 6A$

$A = 1$

Sub A, B, C in eq'n (1)

$(1) \Rightarrow f(z) = 1 + \dfrac{3}{(z+2)} - \dfrac{8}{(z+3)}$

i) $|z|>3$

$|z|>3$

$3<|z|$

$$\frac{3}{|z|} < 1$$

$$f(z) = 1 + \frac{3}{(z+2)} - \frac{8}{(z+3)}$$

$$f(z) = 1 + \frac{3}{z(1+2/z)} - \frac{8}{z(1+3/z)}$$

$$f(z) = 1 + 3/z\,(1+2/z)^{-1} - 8/z\,(1+3/z)^{-1}$$

$$f(z) = 1 + 3/z\,(1 - 2/z + (2/z)^2 + \dots) - 8/z\,(1 - 3/z + (3/z)^2 + \dots)$$

$$f(z) = 1 + 3/z \sum_{n=0}^{\infty} (-1)^n (2/z)^n - 8/z \sum_{n=0}^{\infty} (-1)^n (3/z)^n$$

3) Find the laurent's series expansion of $\frac{z-1}{(z+2)(z+3)}$ valued in the region $2<|z|<3$.

$$= \frac{A}{z+2} + \frac{B}{z+3} \rightarrow (1)$$

$$= \frac{A(z+3)+B(z+2)}{(z+2)(z+3)}$$

$$z-1 = A(z+3)+B(z+2)$$

put $z=-2$,

$$-2-1 = A(-2+3)+0$$

$$-3 = A$$

$$\boxed{A=-3}$$

put $z=-3$

$$-3-1 = 0+B(-3+2)$$

$$-4 = -B$$

$$\boxed{B=4}$$

i) $f(z) = \frac{-3}{z+2} + \frac{4}{z+3}$

Given $2<|z|<3$

$2<|z|$ and $|z|<3$

$\frac{2}{|z|}<1$ $\quad \frac{|z|}{3}<1$

$$f(z) = \frac{-3}{z\left(1+\frac{2}{z}\right)} + \frac{4}{3\left(\frac{z}{3}+1\right)}$$

$$= -\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} + \frac{4}{3}\left(\frac{z}{3}+1\right)^{-1}$$

$$f(z) = -\frac{3}{z}\left(1-\frac{2}{z}+\left(\frac{2}{z}\right)^{2}\cdots\right)+\frac{4}{3}\left(1-\frac{z}{3}+\left(\frac{z}{3}\right)^{2}\cdots\right)$$

$$f(z) = \frac{-3}{z}\sum_{n=0}^{\infty}(-1)^{n}\left(\frac{2}{z}\right)^{n}+\frac{4}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^{n}(-1)^{n}$$