Introduction:

If x and y are numbers then $z = x+iy$ is called a complex number where x is called real part of z, y is called the imaginary part of z and the value of i is $\sqrt{-1}$

The complex number $x-iy$ is called as the complex conjugate of z & it is denoted by $\bar{z}$

ie) $\bar{z} = x - iy$

Note:

1. $|z| = \sqrt{x^2+y^2}$
2. $|z^2| = z\bar{z}$
3. $z\bar{z} = x^2+y^2 = r^2$
4. $|\bar{z}| = |z|$
5. Real part of $z = \frac{z+\bar{z}}{2}$
6. Imaginary part of $z = \frac{z-\bar{z}}{2i}$
7. $z = re^{i\theta}$ is called polar form of z
8. Amplitude of $z = \theta = \tan^{-1}(y/x)$

Functions of Complex Variable

$w = f(z) = u(x,y) + iv(x,y)$ where $u(x,y)$ and $v(x,y)$ are real variables

Single Valued function:-

If for each value of z in R there will be only one value of ω, then ω is called a single valued function of z

Eg: $\omega = z^2$, $\omega = 1/z$

$\omega = z^2$

z:	1	2	-2	3
ω:	1	4	4	9

$\omega = 1/z$

z:	1	2	-2	3
ω:	1	1/2	-1/2	1/3

Multiple valued function:

If there is more than one value of ω corresponding to a given value of z, then ω is called a multiple-valued function

Eg: $\omega = z^{1/2}$

z:	4	9	1
ω	-2, 2	-3, 3	1, -1

Analytic function:

A function $f(z)$ is said to be analytic at a point $z=a$ in a region R if

i) $f(z)$ is differentiable at $z=a$

ii) $f(z)$ is differentiable at all points for some neighbourhood of $z=a$

(or)

A function is said to be analytic at a point if its derivative exists not only at the point but also in some neighbourhood of that point

Cauchy-Riemann Equations (Cartesian Coordinates):

Necessary Condition:

If the function $f(z) = u(x,y) + iv(x,y)$ is analytic in a region R of the z plane, then

i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exists

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ ~~(or) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$~~

(or)

at every point in that region.

$u_x = v_y$, $v_x = -u_y$

Sufficient Conditions:

If the function $f(z) = u(x,y) + iv(x,y)$ is analytic in a Region R of the z-plane if

i) u_x, u_y, v_x & v_y exist and all are continuous

(ii) $u_x = v_y$ and $u_y = -v_x$

Cauchy-Riemann Equations (polar coordinates)

Necessary condition:

If the function $w = f(z) = u(r,\theta) + iv(r,\theta)$ is analytic in a region R of the z-plane then

i) if $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}$ and $\frac{\partial v}{\partial \theta}$ exist

(ii) $\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}}$

Sufficient condition:

If the function $w = f(z) = u(r,\theta) + iv(r,\theta)$ is analytic in a region R of the z-plane, then

i) $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}$ and $\frac{\partial v}{\partial \theta}$ exists and all are continuous

(ii) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$