Problems :

1. Prove that $w = z^2$ is analytic.

We know that $z = x+iy$

$w = z^2 = (x+iy)^2 = x^2 + 2ixy - y^2$

$u+iv = x^2 - y^2 + 2ixy$

$u = x^2 - y^2$; $v = 2xy$

$u_x = 2x$ $\quad$ $v_x = 2y$

$u_y = -2y$ $\quad$ $v_y = 2x$

$u_x = v_y$ & $u_y = -v_x$

$\therefore$ It satisfies CR equ

$\Rightarrow$ $w = z^2$ is analytic

2. Determine whether the function $w = 2xy + i(x^2 - y^2)$ is analytic.

Given: $w = 2xy + i(x^2 - y^2)$

$u+iv = 2xy + i(x^2 - y^2)$

$u = 2xy$ $\quad$ $v = x^2 - y^2$

$u_x = 2y$ $\quad$ $v_x = 2x$

$u_y = 2x$ $\quad$ $v_y = -2y$

$u_x \neq v_y$ & $u_y \neq -v_x$

It does not satisfies CR equations

$\Rightarrow$ $w = 2xy + i(x^2 - y^2)$ is not analytic.

③ Verify whether $f(z) = \sinh z$ is analytic using CR eqns

$f(z) = \sinh z$ $\qquad\qquad$ $\sin i\theta = i \sinh\theta$
$\qquad\qquad\qquad\qquad$ $\cos i\theta = \cosh\theta$

$u+iv = \sinh(x+iy)$

$= \frac{1}{i} \sin i(x+iy)$ [Multiply & divide by i]

$= \frac{1}{i} \sin(ix+i^2y) = \frac{1}{i}\sin(ix-y)$

$= \frac{1}{i}[\sin ix \cos y - \cos ix \sin y]$

$= \frac{1}{i}[i\sinh x \cos y - \cosh x \sin y]$

$= \sinh x \cos y - \frac{1}{i}\cosh x \sin y$ $\qquad$ $[\frac{1}{i} = -i]$

$= \sinh x \cos y + i \cosh x \sin y$

$u = \sinh x \cos y$; $v = \cosh x \sin y$

$u_x = \cosh x \cos y$; $v_x = \sinh x \sin y$

$u_y = -\sinh x \sin y$ $\qquad$ $v_y = \cosh x \cos y$

$\Rightarrow u_x = v_y$ & $v_x = -u_y$

$\therefore$ It satisfies CR equations

$\therefore f(z) = \sinh z$ is analytic

④ Show that $f(z) = |z|^2$ is nowhere analytic

$f(z) = |z|^2 \Rightarrow u+iv = x^2+y^2$

$u = x^2+y^2$, $v=0$

$u_x = 2x$ $\qquad$ $v_x = 0$

$u_y = 2y$ $\qquad$ $v_y = 0.$

$u_x \neq v_y$ & $u_y \neq -v_x$

It doesn't satisfies CR eqns

$\therefore$ It is nowhere analytic

$f(z) = |z|^2$

(5) If $u+iv$ is analytic then $v-iu$ is also analytic.

$u+iv$ is analytic

ie) CR eqns are satisfied

ie) $u_x = v_y$ & $u_y = -v_x$

ie) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

To prove: $v-iu$ is also analytic

ie., we have to prove, $\frac{\partial v}{\partial x} = \frac{\partial(-u)}{\partial y}$ & $\frac{\partial v}{\partial y} = -\frac{\partial(-u)}{\partial x}$

$$\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y} \quad \& \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

wkt, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$

Hence $v-iu$ is also analytic

(6) If $w = e^z$, find $\frac{dw}{dz}$ using complex variable

$w = e^z$

$u+iv = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$

$= e^x\cos y + ie^x\sin y$

$u = e^x\cos y \quad ; \quad v = e^x \sin y$

$u_x = e^x\cos y \qquad v_x = e^x\sin y$

$u_y = -e^x\sin y \qquad v_y = e^x\cos y$

[Result:- If $w = f(z) = u+iv$ then

$\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

finding $\frac{dw}{dz}$ in terms of partial derivatives w.r.t x]

$$\Rightarrow \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= e^x \cos y + i e^x \sin y$$

$$= e^x(\cos y + i \sin y)$$

$$= e^x e^{iy}$$

$$= e^{x+iy}$$

$$= e^z.$$

7) State whether the function $\sin(x-iy)$ is analytic (or) not. Justify your answer

Given, $f(z) = \sin(x-iy)$ $\qquad (\because \sin(A-B) = \sin A\cos B - \cos A \sin B$

$$u+iv = \sin x \cos iy - \cos x \sin iy$$

$\because \cosh y = \cos iy$

Replace, $\qquad$ |||ly $\sin iy = -i\sinh y$

$$u+iv = \sin x\cosh y - \cos x(-i\sinh y)$$

$$= \sin x\cosh y + i\cos x \sinh y$$

$u = \sin x \cosh y \qquad v = \cos x \sinh y$

$\frac{\partial u}{\partial x} = \cos x\cosh y \qquad \frac{\partial v}{\partial x} = -\sin x \sinh y$

$\frac{\partial u}{\partial y} = \sin x\sinh y \qquad \frac{\partial v}{\partial y} = \cos x \cosh y.$

$u_x = v_y = \cos x\cosh y$

$v_x = -u_y = \sin x \sinh y$

$\therefore$ It satisfies CR equations

$\therefore \sin(x-iy)$ is analytic

8.. Verify whether the function $f(z) = e^{-x}(\cos y - i\sin y)$ is analytic. Justify your answer. Analytic

$f(z) = e^{-x}(\cos y - i\sin y)$

$u+iv = e^{-x}\cos y - i\,e^{-x}\sin y$

$u = e^{-x}\cos y \qquad v = -e^{-x}\sin y$

$\frac{\partial u}{\partial x} = -e^{-x}\cos y \qquad \frac{\partial v}{\partial x} = e^{-x}\cos y$

$\frac{\partial u}{\partial y} = -e^{-x}\sin y \qquad \frac{\partial v}{\partial y} = -e^{-x}\cos y$

$u_x = v_y \;\&\; u_y = -v_x$

∴ It satisfies CR equations

$\Rightarrow f(z) = e^{-x}(\cos \cancel{y} - i\sin y)$ is analytic.

9. $f(z) = z^3$ is analytic or not.

$f(z) = z^3 = (x+iy)^3 = x^3 + i^3y^3 + 3xiy(x+iy)$

$= x^3 - iy^3 + i\,3x^2y - 3xy^2$

$u = x^3 - 3xy^2 \qquad v = 3x^2y - y^3$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial v}{\partial x} = 6xy$

$\frac{\partial u}{\partial y} = -6xy \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$

$u_x = v_y \;\&\; v_x = -u_y$

∴ It satisfies CR Equations

$\Rightarrow f(z) = z^3$ is analytic.