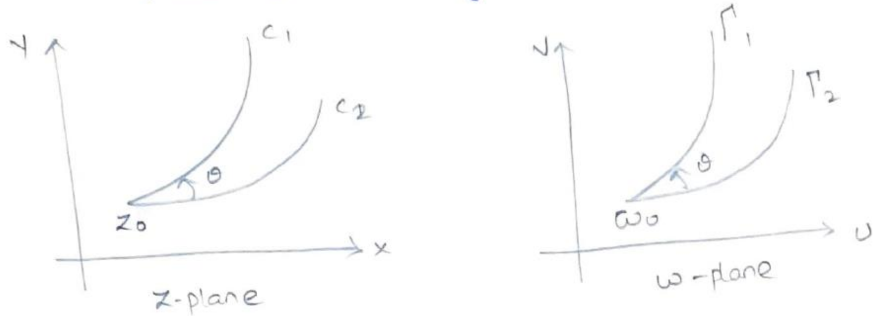
Conformal Mapping

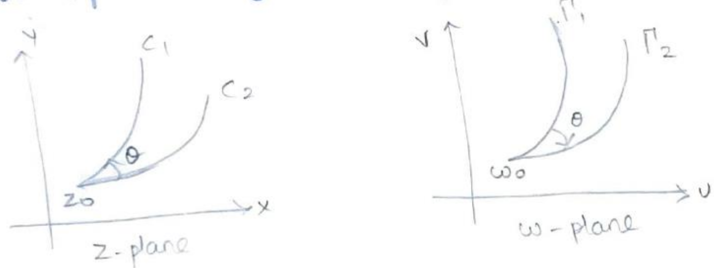
Definition :

A mapping $w=f(z)$ is said to be conformal at $z=z_0$ if it preserves the angle between any two curves through z_0 in z plane both in magnitude and direction



Isogonal mapping :

A mapping $w=f(z)$ is said to be isogonal at $z=z_0$ if it preserves the angle between any two curves through z_0 in z plane only in magnitude but not in direction



Remarks:

1. If $f(z)$ is analytic and $f'(z) \neq 0$ at each point then the mapping $w=f(z)$ is conformal.
2. The points at which $w=f(z)$ is not conformal. ie., $f'(z)=0$ are called critical points
3. The critical points of $w=f(z)$ will occur at

$$\frac{dz}{dw}=0 \text{ and } \frac{dw}{dz}=0$$

① Find the critical points of the transformation $w = z + \frac{1}{z}$

$$\frac{dw}{dz} = 1 - \frac{1}{z^2} = \frac{z^2-1}{z^2}$$

$$\frac{dz}{dw} = \frac{z^2}{z^2-1}$$

The critical points occur at $\frac{dw}{dz} = 0$, $\frac{dz}{dw} = 0$

$\Rightarrow \frac{z^2-1}{z^2} = 0$ and $\frac{z^2}{z^2-1} = 0$

$\Rightarrow z^2 - 1 = 0$ and $z^2 = 0$

$z^2 = 1$ $\qquad z = 0$

$z = \pm 1$

$\therefore z = 0, 1, -1$ are the critical points

② Find the critical points of $w^2 = (z-\alpha)(z-\beta)$

$$w^2 = (z-\alpha)(z-\beta)$$

$$2w\frac{dw}{dz} = (z-\alpha) + (z-\beta)$$

$$\frac{dw}{dz} = \frac{(z-\alpha)+(z-\beta)}{2w}$$

$$\frac{dz}{dw} = \frac{2w}{(z-\alpha)+(z-\beta)}$$

$\therefore \frac{dw}{dz} = \frac{(z-\alpha)+(z-\beta)}{2w}$ and $\frac{dz}{dw} = \frac{2w}{(z-\alpha)+(z-\beta)}$

The critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$

$\Rightarrow \frac{dw}{dz} = 0 \quad \Rightarrow \frac{(z-\alpha)+(z-\beta)}{2w} = 0$

$\Rightarrow z-\alpha+z-\beta=0$

$2z=\alpha+\beta$

$z=\frac{\alpha+\beta}{2}$

$\Rightarrow \frac{dz}{dw}=0$ $\quad\Rightarrow \frac{2w}{(z-\alpha)+(z-\beta)}=0$

$\Rightarrow 2w=0$

$\Rightarrow w=0$

$\Rightarrow w^2=0.$

$\Rightarrow (z-\alpha)(z-\beta)=0$

$z=\alpha,\ z=\beta$

$\therefore z=\frac{\alpha+\beta}{2}, \alpha, \beta$ are the critical points

③ Find the points such that $w=f(z)=\sin z$ is not conformal.

Let $w=\sin z$

$\frac{dw}{dz}=\cos z$; $\frac{dz}{dw}=\frac{1}{\cos z}$

The critical points ~~occur~~ occur at $\frac{dw}{dz}=0$ and $\frac{dw}{dz}=0$

$\Rightarrow \cos z=0$ and $\frac{1}{\cos z}=0$

$z=\cos^{-1}(0)$ and $1=0$, impossible

$z=\pm(2n-1)\frac{\pi}{2}$, $n=0,1,2,\ldots$