Laurents' Series

Let c_1 and c_2 be two concentric circles $|z-a|=R_1$ and $|z-a|=R_2$ where $R_2<R_1$.

Let $f(z)$ be analytic inside and on the annular region R between c_1 and c_2. Then for any $z \in R$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

where $$a_n = \frac{1}{2\pi i} \int_{c_1} \frac{f(z)}{(z-a)^{n+1}}\, dz$$

$$b_n = \frac{1}{2\pi i} \int_{c_2} \frac{f(z)}{(z-a)^{1-n}}\, dz.$$

Problems:

① Expand $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ in a laurent's series if i) $|z|<2$ ii) $|z|>3$ and (iii) $2<|z|<3$.

Solution:- Using partial fractions

$$f(z) = \frac{z^2-1}{(z+2)(z+3)} = A + \frac{B}{z+2} + \frac{C}{z+3} \rightarrow ①$$

$$\frac{z^2-1}{(z+2)(z+3)} = \frac{A(z+2)(z+3) + B(z+3) + C(z+2)}{(z+2)(z+3)}$$

$$z^2-1 = A(z+2)(z+3) + B(z+3) + C(z+2)$$

Put $z=-2$

$$(-2)^2-1 = A(0) + B(-2+3) + C(0)$$

$$4-1 = B$$

$$\boxed{B=3}$$

Put $z = -3$

$(-3)^2 - 1 = A(0) + B(0) + C(-3+2)$

$9 - 1 = -C$

$\boxed{C = -8}$

Put $z = 0$.

$(0)^2 - 1 = A(0+3)(0+2) + B(0+3) + C(0+2)$

$-1 = A(3)(2) + B(3) + C(2)$

$-1 = 6A + 3B + 2C$

Sub $B = 3$ & $C = -8$ in

$-1 = 6A + 3(3) + 2(-8)$

$-1 = 6A + 9 + (-16)$

$-1 = 6A - 7$

$6A = -1 + 7$

$6A = 6$

$\boxed{A = 1}$

$① \Rightarrow \quad f(z) = 1 + \frac{3}{(z+2)} + \frac{(-8)}{(z+3)} \rightarrow ②$.

(i) $|z| < 2$

$\Rightarrow \frac{|z|}{2} < 1$

$② \Rightarrow \quad f(z) = 1 + \frac{3}{2(\frac{z}{2}+1)} - \frac{8}{3(1+\frac{z}{3})}$

$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$

$$= 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 \dots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \dots\right]$$

$$\left[\because (1+z)^{-1} = 1 - z + z^2 - z^3 + \dots\right]$$

(ii) $|z| > 3$

$3 < |z|$

$\frac{3}{|z|} < 1$

② $\Rightarrow f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$

$$= 1 + \frac{3}{z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1 + \frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \dots\right] - \frac{8}{z}\left[1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \dots\right]$$

(iii) $2 < |z| < 3$

$2 < |z|$ and $|z| < 3$

$\frac{2}{|z|} < 1$ and $\frac{|z|}{3} < 1$

② $\Rightarrow = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$

$$= 1 + \frac{3}{z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1 + \frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \dots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \dots\right]$$

(2) Expand $f(z) = \frac{7z-2}{z(z-2)(z+1)}$ in laurents series if

i) $|z|<2$ (ii) $|z|>3$ (iii) $2<|z|<3$ iv) $1<|z+1|<3$

Soln:

Given: $f(z) = \frac{7z-2}{z(z-2)(z+1)}$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{(z-2)} + \frac{C}{(z+1)}$$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A(z-2)(z+1) + B(z)(z+1) + C(z)(z-2)}{z(z-2)(z+1)}$$

$$7z-2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$$

Put $z=0$

$$7(0)-2 = A(-2)(1) + B(0) + C(0)$$

$$-2 = -2A \quad \boxed{A=1}$$

Put $z=2$

$$7(2)-2 = A(0) + B(2)(2+1) + C(0)$$

$$14-2 = B(6)$$

$$12 = 6B$$

$$\boxed{B=2}$$

Put $z=-1$

$$7(-1)-2 = A(0) + B(0) + C(-1)(-1-2)$$

$$-7-2 = 3C$$

$$-9 = 3C \quad \boxed{C=-3} \quad \Rightarrow f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

i) $|z|<2 \Rightarrow \frac{|z|}{2}<1$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{z} + \frac{2}{-2(1-\frac{z}{2})} - \frac{3}{z+1}$$

$$= \frac{1}{z} - (1-\frac{z}{2})^{-1} - 3(z+1)^{-1}$$

$$= \frac{1}{z} - \left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{3}\right)^3+\cdots\right] - 3\left[1-z+z^2-z^3+\cdots\right]$$

(ii) $|z|>3 \Rightarrow \frac{3}{|z|}<1$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{z} + \frac{2}{z(1-\frac{2}{z})} - \frac{3}{z(1+\frac{1}{z})}$$

$$= \frac{1}{z} + \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{3}{z}\left(1+\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z} + \frac{2}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right] - \frac{3}{z}\left[1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\cdots\right]$$

(iii) $2<|z|<3$

$|z|>2$, $|z|<3$

$\Rightarrow \frac{2}{|z|}<1$, $\frac{|z|}{3}<1$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{z} + \frac{2}{z(1-\frac{2}{z})} - \frac{3}{z(1+\frac{1}{z})} = \frac{1}{z} + \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{3}{z}\left(1+\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z} + \frac{2}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right] - \frac{3}{z}\left[1-\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\cdots\right]$$

iv) $1 < |z+1| < 3$

Let $u = z+1 \Rightarrow z = u-1$

$$1 < |u| < 3$$

$$\Rightarrow \frac{1}{|u|} < 1 \, , \quad \frac{|u|}{3} < 1$$

$$f(z) = \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u}$$

$$= \frac{1}{u\left(1-\frac{1}{u}\right)} + \frac{2}{(-3)\left(1-\frac{u}{3}\right)} - \frac{3}{u}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1} - \frac{3}{u}$$

$$= \frac{1}{u}\left(1+\frac{1}{u}+\left(\frac{1}{u}\right)^2+\dots\right) - \frac{2}{3}\left(1+\frac{u}{3}+\left(\frac{u}{3}\right)^2+\left(\frac{u}{3}\right)^3+\dots\right) - \frac{3}{u}$$

$$f(z) = \frac{1}{z+1}\left[1+\left(\frac{1}{z+1}\right)+\left(\frac{1}{z+1}\right)^2+\dots\right] - \frac{2}{3}\left[1+\left(\frac{z+1}{3}\right)+\left(\frac{z+1}{3}\right)^2+\left(\frac{z+1}{3}\right)^3+\dots\right] - \frac{3}{z+1}$$