Problems

1. Evaluate $\int_C \frac{z^2 - z+1}{z-1} dz$ where C is the circle.

i) $|z|=1$ ii) $|z| = \frac{1}{2}$ using Cauchy's integral formula.

Given: $\int_C \frac{z^2-z+1}{z-1} dz$

Formula: $\int_C \frac{f(z)}{z-a} dz = 2\pi i \, f(a)$

closed curve C:

i) $|z|=1$

$z = a = 1$

$|1| = 1 \Rightarrow 1 = 1$

$f(z) = z^2 - z + 1$

$z = a = 1$

$f(a) = f(1) = (1)^2 - 1 + 1 = 1$

$f(a) = 1$

1 lies on the circle

$\therefore \int_C \frac{z^2-z+1}{z-1} dz = 2\pi i \, f(a)$

$= 2\pi i (1)$

$= 2\pi i$

ii) $|z| = \frac{1}{2}$

$z = a = 1$

$|1| = \frac{1}{2}$

ie) $1 > \frac{1}{2}$

$z = a = 1$ lies outside the given circle $|z| = \frac{1}{2}$

$\therefore \int_C \frac{z^2-z+1}{z-1} dz = 0$.

2. Evaluate $\int_C \frac{2}{(z-1)(z+3)} dz$ where C is the circle $|z-1| = 2$ using Cauchy's Integral formula.

Given $\int_C \frac{2}{(z-1)(z+3)} dz$

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i\, f(a)$$

closed curve C,

$|z-1| = 2$

$z = a = 1$

$|1-1| = 2$

$0 < 2$

$\therefore$ lies inside the circle

Case 1: $f(z) = \frac{2}{z+3}$

$z = a = 1$

$f(a) = f(1) = \frac{2}{1+3} = \frac{2}{4} = \frac{1}{2}$

$$\int_C \frac{2/z+3}{z-1} dz = 2\pi i\,(1/2)$$

$= \pi i$

Case 2: $z = a = -3$

$|z-1| = 2$

$|-3-1| = 2 \quad \Rightarrow |-4| = 2$

ie, $4 > 2$

point lies outside the circle

$$\int \frac{2}{(z-1)(z+3)} dz = 0$$

$$\therefore \int_C \frac{2}{(z-1)(z+3)} dz = \pi i + 0 = \pi i$$