Singular Points:

A point $z=a$ is said to be a singular point or singularity of $f(z)$ if $f(z)$ is not analytic at $z=a$

Types of Singular points:

1. Isolated Singular point:

A point $z=a$ is said to be an isolated singular point of $f(z)$ if

i) $f(z)$ is not analytic at $z=a$ (z_0)

ii) $f(z)$ is analytic at all points for some neighbourhood of $z=a$ ∃ a neighbourhood of $z=z_0$ containing no other singularity

Example: $f(z) = \frac{z}{(z-1)(z-2)}$

$z=1$ & $z=2$ are isolated singular points.

2. pole:

A point $z=a$ is said to be a pole of order n if we can find a positive integer n such that

$$\lim_{z \to a} (z-a)^n f(z) \neq 0.$$

Example: $f(z) = \frac{z-1}{(z-2)(z-3)^4}$

$z=2$ is a pole of order 1 (or) simple pole.

$z=3$ is a pole of order 4.

3. Essential Singularity:

If the principal part of $f(z)$ in its laurents series does not terminate, ie., it possesses infinite number of terms, then $z=a$ is called an essential singular point of $f(z)$

Example: $f(z) = e^{1/z}$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^8} + \dots \infty$$

$\therefore$ $z=0$ is an essential singularity.

4. Removable Singularity :-

The singular point $z=a$ is called a removable singularity of $f(z)$ if $\underset{z \to a}{Lt} f(z)$ exists

Example :- $f(z) = \frac{\sin z}{z}$

$z=0$ is the singular point

$$\therefore \underset{z \to a}{Lt} f(z) = \underset{z \to 0}{Lt} \frac{\sin z}{z}$$

$$= \underset{z \to 0}{Lt} \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots}{z}$$

$$= \underset{z \to 0}{Lt} \frac{z\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \dots\right]}{z}$$

$$= \underset{z \to 0}{Lt} \left[1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \dots\right]$$

$$= 1$$

$\therefore$ $\underset{z \to a}{Lt} f(z)$ exist. It is a removable singularity