Harmonic Function

If Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ or

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$ is satisfied

ie) $U_{xx} + U_{yy} = 0$ (or) $V_{xx} + V_{yy} = 0.$

ie) $f(z)$ is said to be harmonic function

(or)

An expression of the form $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$ is called the laplace equation in two dimension

Any function having continuous second order partial derivatives which satisfies the laplace equation is called harmonic function.

Any two harmonic functions u and v such that $f(z) = u + iv$ is analytic are called conjugate harmonic functions.

Note:

Both real and imaginary parts of an analytic function are harmonic. But the converse need not be true.

① P.T $U = x^4 - 6x^2y^2 + y^4$ harmonic function.

Harmonic equation : $U_{xx} + U_{yy} = 0$

$$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2 \qquad \frac{\partial^2 u}{\partial x^2} = 12x^2 - 12y^2$$

$$\frac{\partial u}{\partial y} = 4y^3 - 12x^2y \qquad \frac{\partial^2 u}{\partial y^2} = 12y^2 - 12x^2$$

$\Rightarrow 12x^2 - 12y^2 + 12y^2 - 12x^2 = 0.$

$\Rightarrow U = x^4 - 6x^2y^2 + y^4$ is Harmonic

2. Prove that $v = x^2 - y^2 + 2xy - 3x - 2y$ is harmonic

$\frac{\partial v}{\partial x} = 2x + 2y - 3 \qquad \frac{\partial v}{\partial y} = -2y + 2x - 2$

$\frac{\partial^2 v}{\partial x^2} = 2 \qquad \frac{\partial^2 v}{\partial y^2} = -2$

Harmonic function:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 2 - 2 = 0.$$

$\therefore v = x^2 - y^2 + 2xy - 3x - 2y$ is Harmonic fn

3. Prove that $u = e^x \cos y$ is harmonic function

$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial^2 u}{\partial y} = -e^x \sin y$

$\frac{\partial^2 u}{\partial y} = e^x \cos y \qquad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$

Harmonic function:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y$$

$$= 0.$$

$\therefore u = e^x \cos y$ is harmonic

4. Find the value of m such that $2x - x^2 + my^2$ is harmonic function

Let $u = 2x - x^2 + my^2$

$$\frac{\partial u}{\partial x} = 2 - 2x \qquad \frac{\partial u}{\partial y} = 2my$$

$$\frac{\partial^2 u}{\partial x^2} = -2 \qquad \frac{\partial^2 u}{\partial y^2} = 2m$$

Harmonic function

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2+2m = 0.$$

$$2m = 2$$

$$\boxed{m = 1}$$

5. P.T $u = x^2 - y^2 - 2xy - 2x + 3y$ is a harmonic function

Given $u = x^2 - y^2 - 2xy - 2x + 3y$

$$\frac{\partial u}{\partial x} = 2x - 2y - 2 \qquad \frac{\partial u}{\partial y} = -2y - 2x + 3$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

Harmonic function :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0.$$

$\therefore u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic

6) Give an example such that u and v are harmonic but u+iv is not analytic

Let $w = \bar{z}$

$u + iv = x - iy$

$\Rightarrow u = x \; ; \; v = -y$

$\frac{\partial u}{\partial x} = 1 \; ; \; \frac{\partial u}{\partial y} = 0 \; ; \; \frac{\partial v}{\partial x} = 0 \; ; \; \frac{\partial v}{\partial y} = -1$

$\frac{\partial^2 u}{\partial x^2} = 0 \; ; \; \frac{\partial^2 u}{\partial y^2} = 0 \quad \frac{\partial^2 v}{\partial x^2} = 0 \quad \frac{\partial^2 v}{\partial y^2} = 0.$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$

$\Rightarrow$ u and v are harmonic

But $u_x \neq v_y$ and $v_x \neq -u_y$

$\therefore f(z) = u + iv$ is not analytic

7) Prove that $u = x^2 - y^2$, $v = \frac{-y}{x^2+y^2}$ are harmonic but u+iv is not a regular function

Let $u = x^2 - y^2$

$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y$

$\frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial^2 u}{\partial y^2} = -2.$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$

$\Rightarrow$ u is harmonic

Let $v = \dfrac{-y}{x^2+y^2}$

$$\frac{\partial v}{\partial x} = \frac{-[(x^2+y^2)(0) - y(2x)]}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)^2(2y) - 2xy \cdot 2(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2)^2 - 8x^2y(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2) - 8x^2y}{(x^2+y^2)^3} = \frac{2y^3+2x^2y-8x^2y}{(x^2+y^2)^3}$$

$$= \frac{2y^3-6x^2y}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{-[(x^2+y^2) - y\cdot 2y]}{(x^2+y^2)^2} = \frac{-(x^2-y^2)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2 2y - (y^2-x^2)\cdot 2(x^2+y^2)2y}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)^2 2y - 4y(y^2-x^2)(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2) - 4y(y^2-x^2)\cancel{(x^2+y^2)}}{(x^2+y^2)^3}$$

$$= \frac{2y^3+2yx^2-4y^3+4x^2y}{(x^3+y^2)^3}$$

$$= \frac{6x^2y-2y^3}{(x^2+y^2)^3}$$

Harmonic function:-

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3} + \frac{6x^2y - 2y^2}{(x^2+y^2)^3}$$

$$= 0$$

$\therefore$ V is harmonic

But $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} \neq \frac{-\partial v}{\partial x}$