Construction of conjugate harmonic fcns :

Method 1: Suppose u is given then

$$v = \int\left(-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy\right) + c$$ where c is a constant

Method 2: Suppose v is given then

$$u = \int\left(\frac{\partial v}{\partial y}\,dx - \frac{\partial v}{\partial x}\,dy\right) + c$$ where c is a constant

① Show that the function $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic and find its harmonic conjugate

Let $u = \frac{1}{2}\log(x^2+y^2)$

$$u_x = \frac{1}{2}\frac{1}{x^2+y^2}\cdot 2x = \frac{x}{x^2+y^2}$$

$$u_{xx} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)}$$

$$u_y = \frac{1}{2}\frac{1}{x^2+y^2}\cdot 2y = \frac{y}{x^2+y^2}$$

$$u_{yy} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$u_{xx} + u_{yy} = 0$$

u satisfies Laplace equation

u is harmonic

$$x\,dy - y\,dx = (x^2+y^2)\,dx$$

$$V = \int\left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right)+c$$

$$\Rightarrow \frac{x\,dy - y\,dx}{(x^2+y^2)} = dx$$

$$= \int\left(\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right)+c$$

$$\frac{\left(\frac{x\,dy - y\,dx}{x^2}\right)}{\left(1+\left(\frac{y}{x}\right)^2\right)} = dx$$

$$= \int \frac{x\,dy - y\,dx}{x^2+y^2} + c = \int \frac{d(y/x)}{1+y^2/x^2} + c \qquad \tan^{-1}\left(\frac{y}{x}\right)+c$$

$$V = \tan^{-1}(y/x) + c$$

2. P.T the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic. Find the harmonic conjugate function.

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

$$u_x = 3x^2 - 3y^2 + 6x \qquad u_y = -6xy - 6y$$

$$u_{xx} = 6x + 6 \qquad u_{yy} = -6x - 6$$

$$u_{xx} + u_{yy} = 6x+6-6x-6 = 0.$$

u satisfies Laplace equation.

$\Rightarrow u$ is harmonic

$$V = \int\left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right)+c$$

$$= \int\left[-(-6xy-6y)\,dx + (3x^2-3y^2+6x)\,dy\right] + c$$

$$= 3x^2y + 6xy + 3x^2y - y^3 + 6xy + c$$
$$= 6x^2y + 12xy - y^3 + c$$

2) $u = 6x^2 - 2x + 3xy^2$
3) $u = 3x^2y + 2x^2 - y^3 - 2y^2$
4) $u = e^x \cos y$

3) S.T $u = \cos x \cosh y$ is harmonic & hence find its harmonic Conjugate

$u = \cos x \cosh y$

$u_x = -\sin x \cosh y$ $\qquad$ $u_y = \cos x \sinh y$

$u_{xx} = -\cos x \cosh y$ $\qquad$ $u_{yy} = \cos x \cosh y$

$u_{xx} + u_{yy} = 0.$

$\Rightarrow u$ is harmonic

$$V = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) + c$$

$$= \int \left[- \cos x \sinh y \, dx + (- \sin x \cosh y) dy \right] + c$$

$$= \int [- \sin x \sinh y - \sin x \sinh y] + c$$

$$V = -2 \sin x \sinh y + c$$