Construction of analytic function

Milne - Thompson Method

Case i) If u is given, then construct $f(z) = u+iv$ as follows

i) find u_x, u_y

ii) Find $u_x(z,0)$, $u_y(z,0)$

iii) Find $f(z) = \int [u_x(z,0)\,dz - i\,u_y(z,0)\,dz]$

Case ii) If v is given, then construct $f(z) = u+iv$ as follows

i) Find v_x, v_y

ii) find $v_x(z,0)$, $v_y(z,0)$

iii) find $f(z) = \int [v_y(z,0)\,dz + i\,v_x(z,0)\,dz]$

Problems

1. Find the analytic for $w = f(z)$ where the imaginary part is given by $v = x^2 - y^2 + 2xy - 3x - 2y$.

$V_x = 2x + 2y - 3$ $\qquad$ $V_y = -2y + 2x - 2$

$V_x(z,0) = 2z - 3$ $\qquad$ $V_y(z,0) = 2z - 2$

$$f(z) = \int V_y(z,0)\,dz + i V_x(z,0)\,dz$$

$$= \int (2z-2)\,dz + i(2z-3)\,dz$$

$$= z^2 - 2z + i(z^2 - 3z)$$

2) S.T $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Also find its analytic function $f(z)$

$u_x = 2x - 2y - 2$ $\qquad$ $u_y = -2y - 2x + 3$

$u_x(z,0) = 2z - 2$ $\qquad$ $u_y(z,0) = -2z + 3$

$$f(z) = \int (2z-2) + i(-2z+3)\,dz$$

$$= (z^2 - 2z) + i(3z - z^2) + C$$

$u_{xx} = 2$; $u_{yy} = -2$

$u_{xx} + u_{yy} = 2 - 2 = 0.$

$\Rightarrow$ u is harmonic

and $f(z) = (z^2 - 2z) + i(3z - z^2) + C$.

3. Find an analytic function whose imaginary part is $v = e^{2x}(y\cos 2y + x\sin 2y)$.

$V = e^{2x} y\cos 2y + e^{2x} x\sin 2y$

$V_x = 2e^{2x} y\cos 2y + x \cdot 2e^{2x}\sin 2y + e^{2x}\sin 2y$

$V_x(z,0) = 0$

$$v_y = e^{2x}\cos 2y - 2y\, e^{2x}\sin 2y + 2x\, e^{2x}\cos 2y$$

$$v_y(z,0) = e^{2z} - 0 + 2z\, e^{2z}$$

$$= e^{2z} + 2z\, e^{2z}$$

$$f(z) = \int (e^{2z} + 2z\, e^{2z}) + i(0)\, dz$$

$$= \frac{e^{2z}}{2} + 2\left\{\frac{z e^{2z}}{2} - \frac{e^{2z}}{2^2}\right\}$$

$\int u\,dv = uv_1 - u'v_2 + u''v_3 \ldots$

$$= \frac{e^{2z}}{2} + z e^{2z} - \frac{e^{2z}}{2} = z e^{2z}$$

$u = z \quad v = e^{2z}$

$u' = 1 \quad v_1 = \frac{e^{2z}}{2}$

$u'' = 0 \quad v_2 = \frac{e^{2z}}{2^2}$

$$f(z) = z e^{2z}$$

(4) If real part $u = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$. Find analytic function

Given:

$$u = \frac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$$

$\because \cosh ay = \dfrac{e^{ay} + e^{-ay}}{2}$

$2\cosh ay = e^{ay} + e^{-ay}$

$$= \frac{2\sin 2x}{2\cosh 2y - 2\cos 2x}$$

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x} \quad \begin{matrix} \rightarrow u \\ \rightarrow v \end{matrix}$$

$\dfrac{vu' - uv'}{v^2}$

$$\frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - (\sin 2x)(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2\cos^2 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2(\cos^2 2x + \sin^2 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\left(\frac{\partial u}{\partial x}\right)_{(z,0)} = \frac{2\cos 2z \cosh 2(0) - 2}{(\cosh 2(0) - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1-\cos 2z)^2} \neq \cancel{\frac{-2}{}}$$

$$= \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{-2}{(1-\cos 2z)} = \frac{-2}{1-(1-2\sin^2 z)}$$

$$= \frac{-2}{1-1+2\sin^2 z} = \frac{-2}{2\sin^2 z} = \frac{-1}{\sin^2 z}$$

$$\left(\frac{\partial u}{\partial x}\right)_{(z,0)} = -\operatorname{cosec}^2 z$$

$$\frac{\partial u}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x\,(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$\left(\frac{\partial u}{\partial y}\right)_{(z,0)} = \frac{-2\sin 2z\,\sinh(2(0))}{(\cosh 2(0) - \cos 2(z))^2} = 0$$

By Milne Thomson Method

$$f(z) = \int\left[\left(\frac{\partial u}{\partial x}\right)_{(z,0)} - i\left(\frac{\partial u}{\partial y}\right)_{(z,0)}\right]dz$$

$$= \int(-\operatorname{cosec}^2 z)\,dz = \cot z + c.$$