Conformal mapping

① Find the image of the following region under the translation $\boxed{w = \frac{1}{z}}$

i) half plane $x > c$ when $c > 0$

ii) the infinite strip $\frac{1}{4} < y < \frac{1}{2}$

iii) the infinite strip $0 < y < \frac{1}{2}$.

Soln:

$$w = \frac{1}{z}$$

$$z = \frac{1}{w}$$

$$x + iy = \frac{1}{u+iv} = \frac{1}{u+iv} \cdot \frac{u-iv}{u-iv}$$

$$x + iy = \frac{u-iv}{u^2+v^2} = \frac{u}{u^2+v^2} - \frac{iv}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2} \qquad y = \frac{-v}{u^2+v^2}$$

i) half plane $x > c$ when $c > 0$

$$x = c$$

$$\frac{u}{u^2+v^2} = c$$

$$u = c(u^2+v^2)$$

$$\frac{u}{c} = u^2+v^2$$

$$u^2 - \frac{u}{c} + v^2 = 0$$

$u^2 = \frac{u}{c}$

$a = u \qquad 2ab = \frac{u}{c}$

$b = \frac{u}{2ac}$

$b = \frac{u}{2uc}$

$b = \frac{1}{2c}$

$$\left(u^2 - \frac{u}{c} + \left(\frac{1}{2c}\right)^2\right) + v^2 - \left(\frac{1}{2c}\right)^2 = 0. \qquad a^2 - 2ab + b^2$$

$$\left(u - \frac{1}{2c}\right)^2 + v^2 = \left(\frac{1}{2c}\right)^2$$

which is a circle with centre $\left(\frac{1}{2c}, 0\right)$ & radius $\frac{1}{2c}$.

ii) the infinite strip $\frac{1}{4} < y < \frac{1}{2}$.

$y = \frac{1}{4}$

$\frac{-v}{u^2+v^2} = \frac{1}{4}$

$-4v = u^2+v^2$

$u^2+v^2+4v=0$

$u^2+(v+2)^2-4=0.$

$u^2+(v+2)^2=4$

which is a equation of circle with centre $(0,-2)$ & radius $r=2$

$y = \frac{1}{2}$

$\frac{-v}{u^2+v^2} = \frac{1}{2}$

$-2v = u^2+v^2$

$u^2+v^2+2v=0$

$u^2+(v+1)^2-1=0.$

$u^2+(v+1)^2=1$

which is a equation of circle with centre $(0,-1)$ and radius $r=1$.

(iii) $0 < y < \frac{1}{2}$.

$y=0$

$\frac{-v}{u^2+v^2} = 0.$

$v=0$

which is a straight line in w-plane.

$y = \frac{1}{2}$

$\frac{-v}{u^2+v^2} = \frac{1}{2}$

$-2v = u^2+v^2$

$u^2+v^2+2v = 0$

$u^2+(v+1)^2 = 1$

which is a equation of circle with centre $(0,-1)$ and radius $r=1$.

Type . 2 $\quad w = c+z$

1. What is the region of the w-plane into which the rectangular region in the z-plane bounded by the lines $x=0$, $y=0$, $x=1$ and $y=2$ is mapped under the transformation $w = z+(2-i)$?

Given: $w = z+(2-i)$

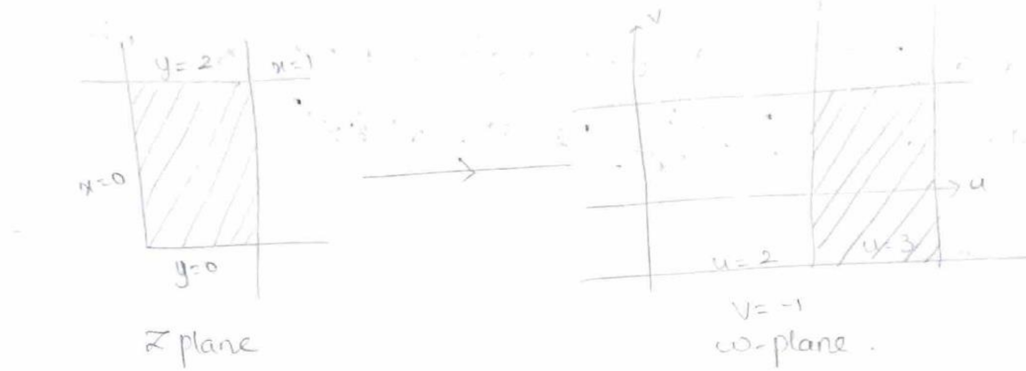
$u+iv = x+iy+(2-i)$

$u+iv = (x+2)+i(y-1)$

Equating real and imaginary parts,

$u = x+2 \quad v = y-1$

Boundary lines	Transformed boundary lines
$x=0$	$u=2$
$y=0$	$v=-1$
$x=1$	$u=3$
$y=2$	$v=1$



Hence the lines $x=0, y=0, x=1$ & $y=2$ in z-plane are transformed into $u=2, v=-1, u=3$ and $v=1$ respectively in ω-plane.

HW Image of $|z-2|=3$, $\omega = z+(3-2i)$

Type 3:

Magnification and Rotation $[\omega = cz]$

1. Find the image of circle $|z|=3$ under the transformation $\omega = 2z$.

Given: $\omega = 2z$

$u+iv = 2(x+iy)$

$u+iv = 2x+2iy$

$\Rightarrow u = 2x$ and $v=2y$

$x = u/2 \qquad y = v/2$

To find the image of $|z|=3$

$|x+iy| = 3$

$\sqrt{x^2+y^2} = 3$

$x^2+y^2 = 9 \Rightarrow \left(\frac{u}{2}\right)^2+\left(\frac{v}{2}\right)^2-9=0$

$\frac{u^2}{4}+\frac{v^2}{4}-9=0$

$u^2+v^2-36=0$

$u^2+v^2=36 \Rightarrow u^2+v^2=6^2$

which represents ~~the~~ a circle with centre at origin & radius 6.

If $f(z) = u+iv$ is a given region function of z in the domain D. P.T $\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right] |f(z)|^2 = 4|f'(z)|^2$.

wkt.

$|f(z)|^2 = f(z)\,(\overline{f(z)})$ $\quad\quad \because z\bar{z} = |z|^2$

$= (u+iv)(u-iv)$

$|f(z)|^2 = u^2-(iv)^2$ ~~for $f(z) = \sqrt{u^2+v^2}$~~

$|f(z)|^2 = u^2+v^2$.

LHS $\Rightarrow$ $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}|f(z)|^2 = \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right] u^2+v^2$

$= \frac{\partial^2}{\partial x^2}(u^2)+\frac{\partial^2}{\partial y^2}(u^2)+\frac{\partial^2}{\partial x^2}(v^2)+\frac{\partial^2}{\partial y^2}(v^2)$

$\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}u^2\right)$

$= 2u\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\left(2.\frac{\partial u}{\partial x}\right)$

$= 2u.\frac{\partial^2 u}{\partial x^2}+2.\left(\frac{\partial u}{\partial x}\right)^2$.

Similarly wrt y

$\frac{\partial^2}{\partial y^2}(u^2) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(u^2)\right) = \frac{\partial}{\partial y}\left(2u.\frac{\partial u}{\partial y}\right)$

$= 2u.\frac{\partial^2 u}{\partial y^2}+2\left(\frac{\partial u}{\partial y}\right)^2$.

$\frac{\partial^2}{\partial y^2}(u^2) = 2u\frac{\partial^2 u}{\partial y^2}+2\left(\frac{\partial u}{\partial y}\right)^2$

Similarly, $\frac{\partial^2}{\partial x^2}(v^2) = 2v\frac{\partial^2 v}{\partial x^2} + 2\left(\frac{\partial v}{\partial x}\right)^2$

$$\frac{\partial^2}{\partial y^2}(v^2) = 2v\frac{\partial^2 v}{\partial y^2} + 2\left(\frac{\partial v}{\partial y}\right)^2$$

$$\frac{\partial^2}{\partial x^2}(u^2) + \frac{\partial^2}{\partial y^2}(u^2) = 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial x}\right)^2 + 2u\frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial u}{\partial y}\right)^2$$

$$= 2u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right)$$

$$= 2u\left[u_{xx} + u_{yy}\right] + 2\left[u_x^2 + u_y^2\right]$$

$$= 2u(0) + 2\left[u_x^2 + (-v_x)^2\right]$$

$$= 2\left[u_x^2 + v_x^2\right]$$

$$= 2|f'(z)|^2$$

Similarly $\frac{\partial^2}{\partial x^2}(v^2) + \frac{\partial^2}{\partial y^2}(v^2) = 2|f'(z)|^2$

LHS $\rightarrow = 2|f'(z)|^2 + 2|f'(z)|^2$

$$= 4|f'(z)|^2$$

Transformation :-

1. Discuss the transformation $w = \frac{1}{z}$ transforms circles and straight line in the z-plane into circles and straight line in the w-plane

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$x+iy = \frac{1}{u+iv} = \frac{1}{u+iv} \times \frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2}$$

$$x+iy = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2} \qquad y = \frac{-v}{u^2+v^2}$$

Consider the equation,

$$a(x^2+y^2) + bx + cy + d = 0 \quad \rightarrow (1)$$

If $a = 0 \Rightarrow$ (1) is a straight line

If $a \neq 0 \Rightarrow$ (1) is a circle.

Sub x & y in (1)

$$a\left[\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2\right] + b\left[\frac{u}{u^2+v^2}\right] + c\left[\frac{-v}{u^2+v^2}\right] + d = 0$$

$$a\left[\frac{u^2+v^2}{(u^2+v^2)^2}\right] + \frac{bu \mp cv}{u^2+v^2} + d = 0$$

$$\frac{a}{u^2+v^2} + \frac{bu - cv}{u^2+v^2} + d = 0$$

$$\frac{a + bu - cv + d(u^2+v^2)}{u^2+v^2} = 0$$

$$du^2 + dv^2 + bu - cv + a = 0 \quad \rightarrow (2)$$

If $d = 0$, (2) represents a straight line

If $d \neq 0$, (2) represents a circle.

Case i) $a = 0$, $d \neq 0$ ① ⇒ straight line ② ⇒ circle

Straight line z plane not passing through the origin maps circle in w-plane not passing through origin

Case (ii) $a = 0$, $d = 0$. ① ⇒ straight line ② ⇒ straight line

Straight line z plane not passing through the origin maps straight line in w-plane not passing through origin

Case iii) $a \neq 0$, $d \neq 0$. ① ⇒ circle, ② ⇒ circle

Circle z-plane not passing through the origin maps circle in wplane not passing through origin

Case (iv) $a \neq 0$, $d = 0$. ① ⇒ circle ② ⇒ straight line

Circle in z-plane not passing through the origin maps straight line w-plane not passing through origin

Find the image of $|z-2i| = 2$ under the translation $w = \frac{1}{z}$

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$x+iy = \frac{1}{u+iv} = \frac{1}{u+iv} \cdot \frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

$$|z-2i| = 2$$

$$|x+iy-2i| = 2 \Rightarrow |x+i(y-2)| = 2$$

$$\sqrt{x^2+(y-2)^2} = 2^2$$

$$x^2+(y-2)^2 = 4$$

It is in the form of the circle with centre $(0,2)$ & $r=2$

$$x^2+y^2+4-4y = 4$$

$$x^2+y^2-4y = 0.$$

Sub x & y $\Rightarrow \left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 - 4\left(\frac{-v}{u^2+v^2}\right) = 0$

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{4v}{u^2+v^2} = 0.$$

$$\frac{u^2+v^2}{(u^2+v^2)^2} + \frac{4v}{u^2+v^2} = 0.$$

$$\frac{1}{u^2+v^2} + \frac{4v}{u^2+v^2} = 0.$$

$$\frac{1+4v}{u^2+v^2} = 0 \Rightarrow 1+4v=0$$

$$\Rightarrow v = \frac{-1}{4}$$

It is a straight line

The circle not passing through the origin in the z-plane transforms into a straight $v = \frac{-1}{4}$ in the W-plane.

Find the image of $|z+2| = 2$ under the translation $w = \frac{1}{z}$

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$x+iy = \frac{1}{u+iv} = \frac{1}{u+iv} \cdot \frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

$$|z+2| = 2$$

$$|x+iy+2| = 2$$

$$|(x+2)+iy| = 2$$

$$\sqrt{(x+2)^2+y^2} = 2$$

$$(x+2)^2 + y^2 = 4$$

It is in the form of circle with centre $(-2,0)$ & $r=2$

$$x^2+4x+4+y^2=4$$

$$x^2+4x+y^2=0. \rightarrow (1)$$

Sub x & y in (1)

$$\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2+4\left(\frac{u}{u^2+v^2}\right)=0$$

$$\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}+\frac{4u}{u^2+v^2}=0$$

$$\frac{u^2+v^2}{(u^2+v^2)^2}+\frac{4u}{u^2+v^2}=0$$

$$\frac{1}{u^2+v^2}+\frac{4u}{u^2+v^2}=0 \quad \Rightarrow \frac{1+4u}{u^2+v^2}=0$$

$$\Rightarrow 1+4u=0.$$

$$\Rightarrow u=\frac{-1}{4}$$

It is a straight line

The circle not passing through the origin in the z-plane transforms to a straight line $u=\frac{-1}{4}$ in the w-plane.