Bilinear Transformation (or) Mobius Transformation

The Transformation $w = \frac{az+b}{cz+d}$, $ad-bc \neq 0$ where a, b, c, d are complex numbers, is called a bilinear transformation.

The bilinear transformation which transforms z_1, z_2, z_3 into w_1, w_2, w_3 is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Componendo & Dividendo Rule:

if $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

① $z = 1, i, -1$ into $w = 2, i, -2$

② $z = 0, 1, \infty$ into $w = -5, -1, 3$

③ $z = 0, 1, \infty$ into $i, 1, -1$

④ $2, i, 0$ into $0, i, 2$

① $z = 1, i, -1$ into $w = 2; i, -2$.

$$\frac{(w-2)(i+2)}{(w+2)(i-2)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{(w-2)}{(w+2)} = \frac{(z-1)(i+1)(i-2)}{(z+1)(i-1)(i+2)}$$

$$= \frac{(z-1)(i^2-2i+i-2)}{(z+1)(i^2-i+2i-2)} = \frac{(z-1)(-1-i-2)}{(z+1)(-1+i-2)}$$

$$\frac{(w-2)}{(w+2)} = \frac{(z-1)(-i-3)}{(z+1)(i-3)} = \frac{(z-1)(i+3)}{(z+1)(3-i)}$$

By componendo- Dividendo Rule

$$\frac{w-2+w+2}{w-2-w-2} = \frac{(z-1)(i+3)+(z+1)(3-i)}{(z-1)(i+3)-(z+1)(3-i)}$$

$$\frac{2w}{-4} = \frac{6z-2i}{2(iz-3)}$$

$$\boxed{w = \frac{-6z+2i}{iz-3}}$$

② $z = 0, 1, \infty$ into $i, 1, -i$

$z_1 = 0$, $z_2 = 1$, $z_3 = \infty$

$w_1 = i$, $w_2 = 1$, $w_3 = -i$

$$\frac{(w-i)(1+i)}{(w+i)(1-i)} = \frac{z-0}{1-0} = \frac{z}{1}$$

$$\frac{(\omega-i)(1+i)}{(\omega+i)(1-i)} \times \frac{(1+i)}{(1+i)} = z$$

$$\frac{(\omega-i)(1+i)^2}{(\omega+i)(1^2-i^2)} = z$$

$$\frac{(\omega-i)(1+2i-1)}{(\omega+i)(1+1)} = z$$

$$\frac{(\omega-i)(2i)}{2(\omega+i)} = z$$

$$(\omega-i)(2i) = 2z(\omega+i)$$

$$\omega i - z\omega - zi - i^2 = 0$$

$$\omega(i - z) = zi - 1$$

$$\boxed{\omega = \frac{iz-1}{i-z}}$$

3) $z_1 = 0,\ z_2 = 1,\ z_3 = \infty$

$\omega_1 = i \quad \omega_2 = 1 \quad \omega_3 = -1$

$$\frac{(\omega-\omega_1)(\omega_2-\omega_3)}{(\omega-\omega_3)(\omega_2-\omega_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(\omega-i)(1+1)}{(\omega+1)(1-i)} = \frac{(z-0)}{1}$$

$$(\omega-i)(2) = z(\omega+1)(1-i)$$

$$2\omega-2i = (z\omega+z)(1-i)$$

$$2\omega-2i = z\omega - z\omega i + z - zi$$

$$2\omega-2i-z\omega+z\omega i-z+zi = 0$$

$$\omega(2-z+zi)+i(z$$

$$2\omega-2i+z\omega(i-1)+z(i-1)=0$$

4) $z_1 = \infty \quad z_2 = i \quad z_3 = 0$

$\omega_1 = 0 \quad \omega_2 = i \quad \omega_3 = \infty$