Taylor's Series:

If $f(z)$ is analytic inside a circle c with centre at $z=a$, then $f(z)$ can be expressed as,

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots + \frac{(z-a)^n}{n!} f^n(a) + \dots$$

which is convergent at every point inside c. This is called Taylor's series of $f(z)$ about $z=a$

Note:

The Taylor's series of $f(z)$ about the point $z=0$ is given by,

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \dots + \frac{z^n}{n!} f^n(0) + \dots$$

① Expand $f(z) = \log(1+z)$ as Taylor's series about $z=0$ if $|z|<1$

$f(z) = \log(1+z)$ $\qquad f(0) = \log 1 = 0.$

$f'(z) = \frac{1}{1+z}$ $\qquad f'(0) = \frac{1}{1} = 1$

$f''(z) = \frac{-1}{(1+z)^2}$ $\qquad f''(0) = \frac{-1}{1} = -1$

$f'''(z) = \frac{2}{(1+z)^3}$ $\qquad f'''(0) = \frac{2}{1} = 2$

Taylor's series about $z=0$ is given by

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \dots$$

$$= 0 + \frac{z}{1!}(1) + \frac{z^2}{2!}(-1) + \frac{z^3}{3!}(2) + \dots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \dots$$

② Expand $f(z) = \cos z$ about $z = \pi/3$ in Taylor's series

$f(z) = \cos z$ $\qquad f(\pi/3) = \cos(\pi/3) = 1/2$

$f'(z) = -\sin z$ $\qquad f'(\pi/3) = -\sin(\pi/3) = -\sqrt{3}/2$

$f''(z) = -\cos z$ $\qquad f''(\pi/3) = -\cos(\pi/3) = -1/2$

$f'''(z) = \sin z$ $\qquad f'''(z) = \sin(\pi/3) = \sqrt{3}/2$

Taylor's series is,

$$f(z) = f(\pi/3) + \frac{(z-\pi/3)}{1!} f'(\pi/3) + \frac{(z-\pi/3)^2}{2!} f''(\pi/3) + \cdots$$

$$= \frac{1}{2} + \frac{(z-\pi/3)}{1!}\left(-\frac{\sqrt{3}}{2}\right) + \frac{(z-\frac{\pi}{3})^2}{2!}\left(-\frac{1}{2}\right) + \frac{(z-\frac{\pi}{3})^3}{3!}\left(\frac{\sqrt{3}}{2}\right) + \cdots$$

$$= \frac{1}{2} - \frac{(z-\pi/3)}{1!}\left(\frac{\sqrt{3}}{2}\right) - \frac{(z-\frac{\pi}{3})^2}{2!}\left(\frac{1}{2}\right) + \frac{(z-\pi/3)^3}{3!}\left(\frac{\sqrt{3}}{2}\right) + \cdots$$

③ Expand $f(z) = \sin z$ in a Taylor's series about $z = \pi/4$

$f(z) = \sin z$ $\qquad f(\pi/4) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f'(z) = \cos z$ $\qquad f'(\pi/4) = \cos\frac{\pi}{4} = 1/\sqrt{2}$

$f''(z) = -\sin z$ $\qquad f''(\pi/4) = -\sin\left(\frac{\pi}{4}\right) = -1/\sqrt{2}$

$f'''(z) = -\cos z$ $\qquad f'''(\pi/4) = -\cos\left(\frac{\pi}{4}\right) = -1/\sqrt{2}$

Taylor's series formula for $z = a$

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + \cdots$$

At $z = \pi/4$

$$f(z) = f\left(\frac{\pi}{4}\right) + \frac{(z-\pi/4)}{1!} f'(\pi/4) + \frac{(z-\pi/4)^2}{2!} f''(\pi/4) + \frac{(z-\pi/4)^3}{3!} f'''(\pi/4) + \cdots$$

$$f(z) = \frac{1}{\sqrt{2}} + \left(z-\frac{\pi}{4}\right)\left(\frac{1}{\sqrt{2}}\right) + \frac{(z-\pi/4)^2}{2}\left(\frac{-1}{\sqrt{2}}\right) + \frac{\left(z-\frac{\pi}{4}\right)^3}{6}\left(\frac{-1}{\sqrt{2}}\right) + \dots$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(z-\frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(z-\frac{\pi}{4}\right)^2 - \frac{1}{6\sqrt{2}}\left(z-\frac{\pi}{4}\right)^3 + \dots$$

4) Expand $f(z) = e^z$ as Taylors series about the point $z=0$

Given $f(z) = e^z$ and $z=0$

Derivatives	Derivatives at $z=0$
$f(z) = e^z$	$f(0) = e^0 = 1$
$f'(z) = e^z$	$f'(0) = e^0 = 1$
$f''(z) = e^z$	$f''(0) = e^0 = 1$
$f'''(z) = e^z$	$f'''(0) = e^0 = 1$

By Taylors series

$$f(z) = f(a) + \frac{(z-a)}{1!}f'(a) + \frac{(z-a)^2}{2!}f''(a) + \dots$$

$$= f(0) + \frac{(z-0)}{1!}f'(0) + \frac{(z-0)^2}{2!}f''(0) + \frac{(z-0)^3}{3!}f'''(0) + \dots$$

$$= 1 + \frac{z}{1!}(1) + \frac{z^2}{2!}(1) + \frac{z^3}{3!}(1) + \dots$$

$$= 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots$$