Residues:

If $z=a$ is an isolated singular point of $f(z)$, we can find the Laurent's series of $f(z)$ about $z=a$.

i.e) $$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

The coefficient b_1 of $\frac{1}{z-a}$ in the Laurent's series of $f(z)$ is called the residue of $f(z)$ at $z=a$

* If $z=a$ is a simple pole of $f(z)$, then

$$\{\text{Res of } f(z)\}_{z=a} = \lim_{z\to a} (z-a) f(z)$$

* If $z=a$ is a pole of order n, then

$$\{\text{Res of } f(z)\}_{z=a} = \lim_{z\to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z)$$

Zeros of an Analytic function:

If a function $f(z)$, analytic in a region R, is zero at a point $z=z_0$ in R, then z_0 is called a zero of $f(z)$.

Simple zero:-

If $f(z)=0$ and $f'(z)\neq 0$, then $z=z_0$ is called a simple zero of $f(z)$ or a zero of the first order.

zero of order n:

If $f(z_0) = f'(z_0) = \dots = f^{n-1}(z_0) = 0$ and $f^n(z_0) \neq 0$ then z_0 is called zero of order n.

1. Find the zeros of $f(z) = \frac{z^2+1}{1-z^2}$

The zeros of $f(z)$ are given by $f(z)=0$

i.e., $f(z) = \frac{z^2+1}{1-z^2} = \frac{(z+i)(z-i)}{1-z^2} = 0$

i.e., $(z+i)(z-i) = 0$

$z = i$ is a simple zero.

$z = -i$ is a simple zero.

2. Find the zeros of $f(z) = \sin \frac{1}{z-a}$.

The zeros are given by $f(z)=0$.

i.e. $\sin \frac{1}{z-a} = 0$

$\frac{1}{z-a} = \sin^{-1}(0)$

$\frac{1}{z-a} = n\pi$, $n = \pm 1, \pm 2, \pm 3, \ldots$

Problems:

(1) Find the zeros and singularities of the function given below:

i) $\frac{\sin z - z}{z^3}$ ii) $\sin\left(\frac{1}{z+1}\right)$ (iii) $\frac{\cot \pi z}{(z-a)^3}$

i) $f(z) = \frac{\sin z - z}{z^3}$

$z=0$ is the singular point

$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\sin z - z}{z^3}$

$$= \underset{z\to 0}{Lt} \frac{\cancel{z} - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots - \cancel{z}}{z^3}$$

$$= \underset{z\to 0}{Lt} \frac{z^3\left[\frac{-1}{3!} + \frac{z^2}{5!} + \cdots\right]}{z^3}$$

$$= \underset{z\to 0}{Lt} \left[\frac{-1}{6} + \frac{z^2}{5!} + \cdots\right]$$

$$= \frac{-1}{6} \neq 0$$

$\therefore \underset{z\to 0}{Lt} f(z)$ exists

$\therefore$ It is a removable singularity

ii) $f(z) = \sin\left(\frac{1}{z+1}\right)$

$$= \frac{1}{z+1} - \frac{\left(\frac{1}{z+1}\right)^3}{3!} + \frac{\left(\frac{1}{z+1}\right)^5}{5!} - \cdots$$

$\therefore$ $z = -1$ is an essential singularity

iii) $f(z) = \frac{\cot \pi z}{(z-a)^3} = \frac{\cos \pi z}{(z-a)^3 \sin \pi z}$

$(z-a)^3 \sin \pi z = 0$; $\sin \pi z = 0$

$(z-a)^3 = 0$; $\sin \pi z = \sin n\pi$

$\pi z = n\pi$

$z = a$ is a pole of order 3 ; $z = n$, where $n = 0, \pm 1, \pm 2, \ldots$

$\therefore$ $z = 0, \pm 1, \pm 2, \ldots$ are simple poles.

2) Consider the function $f(z) = \frac{\sin z}{z^4}$ find the pole.

Soln: $f(z) = \frac{\sin z}{z^4} = \frac{1}{z^4}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]$

$$= \frac{1}{z^3} - \frac{1}{z \cdot 3!} + \frac{z}{5!} - \cdots$$

$\therefore$ $z = 0$ is a pole of order 3.

(3) Find the residue for $f(z) = \frac{1}{(z+1)(z-2)^2}$

Soln: $f(z) = \frac{1}{(z+1)(z-2)^2}$

Poles: $(z+1)(z-2)^2 = 0$

$z+1 = 0$

$z = -1$

$(z-2)^2 = 0$

$z = 2$ is a pole of order 2.

$z = -1$ is a simple pole.

$$[\text{Res } f(z)]_{z\to a} = \underset{z\to a}{Lt}\ (z-a) f(z)$$

$$[\text{Res } f(z)]_{z\to -1} = \underset{z\to -1}{Lt}\ (z+1)\frac{1}{(z+1)(z-2)^2} = \underset{z\to -1}{Lt}\ \frac{1}{(z-2)^2}$$

$$= \frac{1}{(-1-2)^2} = \frac{1}{9}$$

$z = 2$ is a pole of order 2.

$$[\text{Res } f(z)]_{z\to a} = \underset{z\to a}{Lt}\ \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}(z-a)^n f(z)$$

$$[\text{Res } f(z)]_{z\to 2} = \underset{z\to 2}{Lt}\ \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}(z-2)^2\frac{1}{(z+1)(z-2)^2}$$

$$= \underset{z\to 2}{Lt}\ \frac{d}{dz}\left(\frac{1}{z+1}\right)$$

$$= \underset{z\to 2}{Lt}\ \left(\frac{-1}{(z+1)^2}\right)$$

$$= \frac{-1}{(2+1)^2} = \frac{-1}{9}$$