Cauchy Residue Theorem.

If $f(z)$ is analytic at all points inside and on a simple closed curve c, expect for a finite number of isolated singularities $z_1, z_2, \ldots, z_n$ inside c, then

$$\int_c f(z)\,dz = 2\pi i \left[\text{sum of the residues of } f(z) \text{ at } z_1, z_2, \ldots z_n\right]$$

1. Evaluate $\int_c \frac{e^2}{(z+2)(z+1)^2}\,dz$, c is the circle $|z| = 3$.

Let $f(z) = \frac{e^2}{(z+2)(z+1)^2}$

The poles of $f(z)$ are given by,

$(z+2)(z+1)^2 = 0$.

$\Rightarrow z = -1$ is a pole of order 2.

$\Rightarrow$ ~~z+1~~ $z = -2$ is a pole of order 1.

Given $|z| = 3$

$\Rightarrow z = -1 \Rightarrow |z| = 1 < 3$, lies inside c.

$\Rightarrow z = -2 \Rightarrow |z| = 2 < 3$, lies inside c.

$$\{\text{Res of } f(z)\}_{z=-2} = \lim_{z\to -2} \cancel{(z+2)} \frac{e^2}{\cancel{(z+2)}(z+1)^2} = \lim_{z\to -2} \frac{e^2}{(z+1)^2}$$

$$= \frac{e^2}{(-2+1)^2} = e^2.$$

$$\{\text{Res of } f(z)\}_{z=-1} = \lim_{z\to -1} \frac{1}{1!} \frac{d}{dz}\left[(z+1)^2 \frac{e^2}{(z+1)^2(z+2)}\right]$$

$$= \lim_{z\to -1} \frac{d}{dz}\left(\frac{e^2}{z+2}\right)$$

$$= \lim_{z\to -1} \frac{-e^2}{(z+2)^2} = \frac{-e^2}{(-1+2)^2} = -e^2.$$

$\therefore$ By Cauchy's Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,[\text{sum of residue of } f(z)]$$

$$\int_C \frac{e^2}{(z+2)(z+1)^2}\,dz = 2\pi i\,[e^2 - e^2] = 0.$$

2. Evaluate $\displaystyle\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$, c is a circle $|z| = \frac{3}{2}$

Let $f(z) = \dfrac{4-3z}{z(z-1)(z-2)}$

The poles of $f(z)$ are given by

$z(z-1)(z-2) = 0$

$\Rightarrow z = 0$ is a pole of order 1

$z = 1$ is a pole of order 1

$z = 2$ is a pole of order 1

Given $|z| = \frac{3}{2}$

$\Rightarrow z = 0 \Rightarrow |z| = 0 < \frac{3}{2}$, lies inside c

$\Rightarrow z = 1 \Rightarrow |z| = 1 < \frac{3}{2}$, lies inside c

$\Rightarrow z = 2 \quad \Rightarrow |z| = 2 > \frac{3}{2}$ lies outside C

$$\{\text{Res } f(z)\}_{z=0} = \lim_{z \to 0} (z-0) \frac{4-3z}{z(z-1)(z-2)}$$

$$= \lim_{z \to 0} \frac{4-3z}{(z-1)(z-2)}$$

$$= \frac{4}{(-1)(-2)} = \frac{4}{2} = 2.$$

$$\{\text{Res } f(z)\}_{z=1} = \lim_{z \to 1} (z-1) \frac{4-3z}{z(z-1)(z-2)}$$

$$= \lim_{z \to 1} \frac{4-3z}{z(z-2)}$$

$$= \frac{4-3(1)}{(1)(1-2)} = \frac{1}{-1} = -1$$

$$\{\text{Res } f(z)\}_{z=2} = 0.$$

$$\therefore \int_C \frac{4-3z}{z(z-1)(z-2)} dz = 2\pi i[2-1+0] = 2\pi i.$$