Transforms of elementary functions

① $L(1) = \frac{1}{s}$ where $s > 0$.

Proof:-

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L(1) = \int_0^\infty e^{-st}.1.dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{-1}{s}(0-1) = \frac{1}{s}$$

$$\boxed{L(1) = \frac{1}{s}}$$

② $L(K) = \frac{K}{s}$

③ $L(t) = \frac{1!}{s^2}$

$$L(t) = \int_0^\infty e^{-st}.t\,dt$$

$$= \left[\frac{t e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^\infty$$

$$\boxed{L(t) = \frac{1!}{s^2}}$$

Bernoulli's formula.

$I = uv_1 - u'v_2 + u''v_3 - \ldots$

$u = t \qquad v = e^{-st}$

$u' = 1 \qquad v_1 = \frac{e^{-st}}{-s}$

$u'' = 0 \qquad v_2 = \frac{-e^{-st}}{s^2}$

④ $\boxed{L(t^2) = \frac{2!}{s^3}}$

(5) $L(t^n) = \dfrac{\Gamma(n+1)}{s^{n+1}}$ if $s>0$ & $n>-1$

$$L(t^n) = \int_0^\infty e^{-st} t^n \, dt$$

put $x = st \Rightarrow dx = s\,dt \Rightarrow \dfrac{dx}{s} = dt$

$$L(t^n) = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \int_0^\infty e^{-x} \frac{x^n}{s^{n+1}} \, dx$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n \, dx.$$

$$\boxed{L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}}$$

(6) $L(e^{at}) = \dfrac{1}{s-a}$ if $s-a>0$.

$$L(e^{at}) = \int_0^\infty e^{-st} e^{at} dt$$

$$= \int_0^\infty e^{-(s-a)t} dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$\boxed{L(e^{at}) = \frac{1}{s-a} \text{ if } s-a>0.}$$

(7) $L(e^{-at}) = \dfrac{1}{s+a}$ if $s+a>0$.

$$L(e^{-at}) = \int_0^\infty e^{-st} e^{-at} dt$$

$$= \int_0^\infty e^{-(s+a)t} dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$L(e^{-at}) = \frac{1}{s+a} \text{ if } s+a>0.$$

(8) To find $L(\cos at)$ & $L(\sin at)$

We know that, $e^{i\theta} = \cos\theta + i\sin\theta$

$$L(e^{iat}) = \frac{1}{s-ia}$$

$$= \frac{1}{s-ia} \cdot \frac{s+ia}{s+ia}$$

$$= \frac{s+ia}{s^2+a^2}$$

$$L(\cos at + i\sin at) = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

Equating Real & Imaginary parts,

$$\boxed{L(\cos at) = \frac{s}{s^2+a^2} \qquad L(\sin at) = \frac{a}{s^2+a^2}.}$$

(9) To find $L(\sinh at)$

$$L[\sinh at] = L\left(\frac{e^{at}-e^{-at}}{2}\right) = \frac{1}{2}L(e^{at}) - \frac{1}{2}L(e^{-at})$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a-s+a}{(s-a)(s+a)}\right] = \frac{1}{2}\left[\frac{2a}{(s-a)(s+a)}\right]$$

$$\boxed{L(\sinh at) = \frac{a}{s^2-a^2}} \quad \text{for } s^2 > a^2.$$

(10) To find $L(\cosh at)$

$$L(\cosh at) = L\left\{\frac{1}{2}\left[e^{at}+e^{-at}\right]\right\} = \frac{1}{2}L(e^{at}) + \frac{1}{2}L(e^{-at})$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\} = \frac{1}{2}\cdot\frac{2s}{s^2-a^2}$$

$$\boxed{L(\cosh at) = \frac{s}{s^2-a^2}} \quad \text{for } s^2 > a^2.$$

Problems:

① Find $L(t^8)$

$$L(t^n) = \frac{n!}{s^{n+1}}$$

$$L(t^8) = \frac{8!}{s^{8+1}} = \frac{40320}{s^9}$$

② Find $L(t+1)^2$

$$L[(t+1)^2] = L[t^2+2t+1]$$

$$= L(t^2) + 2L(t) + L(1)$$

$$= \frac{2!}{s^3} + 2\frac{1!}{s^2} + \frac{1}{s}$$

$$= \frac{2!}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

③ Find $L\left(\frac{1}{\sqrt{t}}\right)$

$$L\left(\frac{1}{\sqrt{t}}\right) = L(t^{-1/2})$$

$$= \frac{\Gamma(-\frac{1}{2}+1)}{s^{-1/2+1}} = \frac{\Gamma\frac{1}{2}}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

$\Gamma(n+1) = n\Gamma n$ & $\Gamma\frac{1}{2} = \sqrt{\pi}$

④ $L(\sqrt{t})$

$$L(\sqrt{t}) = L(t^{1/2}) = \frac{\Gamma(\frac{1}{2}+1)}{s^{1/2+1}} = \frac{\frac{1}{2}\Gamma\frac{1}{2}}{s\sqrt{s}} = \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}}$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}$$

⑤ $L(e^t) \Rightarrow$

$$L(e^t) = \frac{1}{s-1}$$

⑤ $L(t^{5/2})$

$$L(t^{5/2}) = \frac{\Gamma(5/2+1)}{s^{5/2+1}} = \frac{5/2\,\Gamma(5/2)}{s^{7/2}}$$

$$= \frac{5/2 \cdot 3/2 \cdot 1/2 \cdot \Gamma(1/2)}{s^{7/2}} = \frac{15/8\sqrt{\pi}}{s^{7/2}}$$

$$= \frac{15\sqrt{\pi}}{8 s^{7/2}}.$$

⑥ $L(e^{5t})$

$$L(e^{at}) = \frac{1}{s-a}$$

$$\Rightarrow L(e^{5t}) = \frac{1}{s-5}$$

⑧ $L(e^{-7t})$

$$L(e^{-7t}) = \frac{1}{s+7}$$

⑨ $L(e^{-t}) \Rightarrow L(e^{-t}) = \frac{1}{s+1}$

⑩ Find $L(\sin 5t)$

$$L(\sin at) = \frac{a}{s^2+a^2}$$

$$L(\sin 5t) = \frac{5}{s^2+25}$$

⑪ Find $L(\cos 6t)$

$$L(\cos at) = \frac{s}{s^2+a^2}$$

$$L(\cos 6t) = \frac{s}{s^2+6^2} = \frac{s}{s^2+36}$$

(12) Find $L(\sin^2 2t)$

$\sin^2 2t = \frac{1-\cos 2t}{2}$

$L(\sin^2 2t) = L\left[\frac{1-\cos 2t}{2}\right]$

$= \frac{1}{2}L[1 - \cos 2(2t)]$

$= \frac{1}{2}\{L(1) - L(\cos 4t)]$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4^2}\right] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$

$= \frac{1}{2}\left[\frac{s^2+16-s^2}{s(s^2+16)}\right]$

$= \frac{8}{s(s^2+16)}$

(13). Find $L(\cos^2 3t)$

$\cos^2 3t = \frac{1+\cos 2t}{2}$

$L[\cos^2 3t] = L\left[\frac{1+\cos 2(3t)}{2}\right]$

$= \frac{1}{2}[L(1) + L(\cos 6t)]$

$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]$

(14) Find $L(\cos^3 2t)$

$\cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$

$L[\cos^3 2t] = \frac{1}{4}L[\cos 3(2t) + 3\cos 2t]$

$= \frac{1}{4}[L(\cos 6t) + 3L(\cos 2t)]$

$= \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{3s}{s^2+4}\right]$

15) Find $L(\sin^3 3t)$

$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

$$L(\sin^3 3t) = L\left[\frac{3\sin 3t - \sin 3(3t)}{4}\right] = \frac{1}{4}\left\{3L(\sin 3t) - L(\sin 9t)\right\}$$

$$= \frac{1}{4}\left[3\left(\frac{3}{s^2+9}\right) - \frac{9}{s^2+81}\right]$$

$$= \frac{1}{4}\left[\frac{9}{s^2+9} - \frac{9}{s^2+81}\right]$$

$$= \frac{9}{4}\left[\frac{1}{s^2+9} - \frac{1}{s^2+81}\right]$$

16) Find $L(\sin 2t \cos 3t)$

$\sin(A+B) + \sin(A-B) = 2\sin A\cos$

$$\sin A\cos B = \frac{\sin(A+B) + \sin(A-B)}{2}$$

$$L(\sin 2t \cos 3t) = L\left[\frac{\sin(2t+3t) + \sin(2t-3t)}{2}\right]$$

$$= L\left[\frac{\sin 5t + \sin(-t)}{2}\right]$$

$$= \frac{1}{2}\left\{L(\sin 5t) - L(\sin t)\right\}$$

$$= \frac{1}{2}\left\{\frac{5}{s^2+25} - \frac{1}{s^2+1}\right\}$$