Properties :-

Change of scale property !

If $L\{f(t)\} = F(s)$, then $L[f(at)] = \frac{1}{a} F(s/a)$

Proof :-

WKt,

$$L[f(at)] = \int_0^\infty e^{-st} f(at) dt$$

put $at = x \Rightarrow t = x/a$

$dt = dx/a$

$$L[f(at)] = \int_0^\infty e^{-s(x/a)} f(x) \frac{dx}{a}.$$

$$= \frac{1}{a} \int_0^\infty e^{-s(x/a)} f(x) dx$$

$$= \frac{1}{a} \int_0^\infty e^{-(s/a)x} f(x) dx$$

$$= \frac{1}{a} \int_0^\infty e^{-(s/a)t} f(t) dt$$

$$= \frac{1}{a} F(s/a)$$

First Shifting property !

If $L\{f(t)\} = F(s)$ then

i) $L[e^{-at} f(t)] = \{L[f(t)]\}_{s \to s+a} = F(s+a)$

ii) $L[e^{at} f(t)] = \{L[f(t)]\}_{s \to s-a} = F(s-a)$

Proof :-

i) WKt, $L[f(t)] = \int_0^\infty e^{-st} f(t) dt = F(s)$

$$L[e^{-at} f(t)] = \int_0^\infty e^{-st} [e^{-at} f(t)] dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)dt$$

$$= F(s+a)$$

(ii) $$L[e^{at} f(t)] = \int_0^\infty e^{-st} [e^{at} f(t)]dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)dt$$

$$= F(s-a)$$

Second shifting Property:-

If $L\{f(t)\} = F(s)$ and $g(t) = \begin{cases} f(t-a), & t>a \\ 0, & t<a \end{cases}$

then $L[g(t)] = e^{-as} F(s)$

Proof:

$$L[g(t)] = \int_0^\infty e^{-st} g(t)dt$$

$$= \int_0^a e^{-st} g(t)\,dt + \int_a^\infty e^{-st} g(t)dt$$

$$L[g(t)] = 0 + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)dt$$

Put $t-a=u \Rightarrow dt = du$

when $t=a \Rightarrow u=0.$

$t\to\infty \Rightarrow u\to\infty$

$$L[g(t)] = \int_0^\infty e^{-s(u+a)} f(u)du$$

$$= \int_0^\infty e^{-us} \cdot e^{-as} f(u)du.$$

$$= e^{-as}\int_0^\infty e^{-us} f(u)du$$

$$= e^{-as}\int_0^\infty e^{-st} f(t)dt \qquad \text{Replace } u\to t$$

$$L[g(t)] = e^{-as} F(s)$$