Laplace transforms of derivatives:

If $L[f(t)] = F(s)$ then $L[f'(t)] = sF(s) - f(0)$

Proof:- $L[f'(t)] = \int_0^\infty e^{-st} f'(t)dt$

Integrating by parts we get,

$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t)(-se^{-st})dt$

$= \left[e^{-\infty} f(\infty) - e^0 f(0)\right] + s\int_0^\infty e^{-st} f(t)\, dt$

$= -f(0) + sL\{f(t)\}$

$= sF(s) - f(0)$

Corollary:-

Let $f''(t) = s^2F(s) - sf(0) - f'(0)$

Let $L[g'(t)] = sG(s) - g(0)$

wkt, $L[f'(t)] = sL[f(t)] - f(0)$

Replace $f(t) \to f'(t)$ & $f'(t) \to f''(t)$ & $f(0) \to f'(0)$

$\Rightarrow L[f''(t)] = sL[f'(t)] - f'(0)$

$= s[s(L[f(t)] - f(0)] - f'(0)$

$= s^2L[f(t)] - sf(0) - f'(0)$

$= s^2F(s) - sf(0) - f'(0).$

Laplace Transform of Integrals :

If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$

Proof:

Let $g(t) = \int_0^t f(t)dt$ and $g(0)=0$ then $g'(t) = f(t)$

WKT, $L[g'(t)] = sL(g(t)) - g(0)$

$= sL(g(t))$

$\Rightarrow L[g(t)] = \frac{1}{s} L[g'(t)]$ $\left\{\because g(t) = \int_0^t f(t)dt,\ g'(t) = f(t)\right.$

$\Rightarrow L\left[\int_0^t f(t)dt\right] = \frac{1}{s} L[f(t)]$

$\Rightarrow L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$

Derivative of Laplace Transform (or) Laplace transform of $tf(t)$

If $L[f(t)] = F(s)$ then $L[t f(t)] = \frac{-d}{ds} F(s)$

Proof:

WKT, $L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)dt$

$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t)dt$

$= \int_0^\infty \frac{\partial}{\partial s} (e^{-st}) f(t)dt$

$= \int_0^\infty -te^{-st} f(t)dt$

$= -\int_0^\infty e^{-st}\, t f(t)dt$

$= -L[t f(t)]$

$\Rightarrow L[tf(t)] = \frac{-d}{ds}[F(s)]$

In General,

$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} [F(s)]$

Problems:

Change of scale property:-

① Find $L[\sinh 3t]$ by using change of scale property

$$L[\sinh t] = \frac{1}{s^2-1} = F(s)$$

$$L[\sinh 3t] = \frac{1}{3} F\left(\frac{s}{3}\right)$$

$$= \frac{1}{3} \frac{1}{\left(\frac{s}{3}\right)^2 - 1}$$

$$= \frac{1}{3} \frac{9}{s^2-9}$$

$$= \frac{3}{s^2-9}$$

② Find $L(\cos 5t)$ using change of scale property?

$$L(\cos t) = \frac{s}{s^2+1} = F(s)$$

$$L(\cos 5t) = \frac{1}{5} F\left(\frac{s}{5}\right)$$

$$= \frac{1}{5}\left[\frac{s/5}{(s/5)^2+1}\right]$$

$$= \frac{1}{5}\left[\frac{5s}{s^2+25}\right]$$

$$= \frac{s}{s^2+25}.$$

③ Given $L[f(t)] = \dfrac{s^2 - s + 1}{(2s+1)^2(s-1)}$ applying the change of scale property show that

$$L[f(2t)] = \frac{s^2 - 2s + 4}{4(s+1)^2(s-2)}$$

Soln:

$$L[f(t)] = \frac{s^2 - s + 1}{(2s+1)^2(s-1)} = F(s)$$

$$L[f(2t)] = \frac{1}{2} F(s/2)$$

$$= \frac{1}{2}\left[\frac{(s/2)^2 - (s/2) + 1}{(2s/2+1)^2 \, (s/2 - 1)}\right]$$

$$= \frac{1}{2}\left[\frac{\frac{s^2 - 2s + 4}{4}}{(s+1)^2 (s-2)/2}\right]$$

$$= \frac{1}{4}\left[\frac{s^2 - 2s + 4}{(s+1)^2 (s-2)}\right]$$

④ Find $L[e^{5t}]$ applying change of scale property

Soln:

$$L(e^{t}) = \frac{1}{s-1} = F(s)$$

$$L(e^{5t}) = \frac{1}{5} F(s/5)$$

$$= \frac{1}{5} \frac{1}{(s/5 - 1)}$$

$$= \frac{1}{5} \frac{5}{s-5}$$

$$= \frac{1}{s-5}$$

First shifting theorem:

① Find $L[e^{-3t}\sin^2 t]$

Proof: $L[e^{-at} f(t)] = F(s+a)$

$$L[e^{-3t}\sin^2 t] = L[\sin^2 t]_{s \to s+3}$$

$$= L\left[\frac{1-\cos 2t}{2}\right]_{s \to s+3}$$

$$= \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}_{s \to s+3}$$

$$= \frac{1}{2}\left\{\frac{1}{s+3} - \frac{s+3}{(s+3)^2+4}\right\}$$

$$= \frac{1}{2}\left\{\frac{4}{(s+3)[(s+3)^2+4]}\right\}$$

$$= \frac{2}{(s+3)[(s+3)^2+4]}$$

② Find $L(t^2 e^{-2t})$

$$L[e^{-at} f(t)] = F(s+a)$$

$$L[e^{-2t} t^2] = [L(t^2)]_{s \to s+2}$$

$$= \left[\frac{2}{s^3}\right]_{s \to s+2}$$

$$= \frac{2}{(s+2)^3}$$

③ Find $L[e^{2t}\cos 5t]$

$$L[e^{2t}\cos 5t] = L[\cos 5t]_{s \to s-2}$$

$$= \left[\frac{s}{s^2+25}\right]_{s\to s-2}$$

$$= \frac{s-2}{(s-2)^2+25}$$

Second Shifting Theorem:

1. Find $L[f(t)]$ where $f(t) = \begin{cases} 0, & 0<t<2 \\ 3 & t>2 \end{cases}$

Soln:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st} f(t)\,dt + \int_2^\infty e^{-st} f(t)\,dt$$

$$= 0 + \int_2^\infty e^{-st}\cdot 3\,dt \quad = 3\int_2^\infty e^{-st}\,dt$$

$$= \frac{-3}{s}\left[e^{-st}\right]_2^\infty \quad = \frac{-3}{s}\left[e^{-\infty} - e^{-2s}\right]$$

$$= \frac{3e^{-2s}}{s}$$

2. Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0<t<\pi \\ 0, & t>\pi \end{cases}$$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\pi e^{-st} f(t)\,dt + \int_\pi^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\pi e^{-st}\sin t\,dt \qquad \left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_0^\pi$$

$$= \frac{-e^{-s\pi}}{s^2+1}(-s\sin\pi - \cos\pi) + \frac{1}{s^2+1}$$

$$= \frac{e^{-s\pi}}{s^2+1} + \frac{1}{s^2+1} = \frac{1+e^{-\pi s}}{s^2+1}$$

Laplace Transforms of Derivatives:

① Find $L[t \sin at]$

$f(t) = t \sin at$

$f'(t) = at \cos at + \sin at$

$f''(t) = a[-at \sin at + \cos at] + a \cos at$

$= 2a \cos at - a^2 t \sin at$.

$f(0) = 0, \; f'(0) = 0$.

$L[f''(t)] = s^2 L[f(t)] - s f(0) - f'(0)$

$L[2a \cos at - a^2 t \sin at] = s^2 L[t \sin at] - s(0) - 0$.

$\Rightarrow 2a L(\cos at) - a^2 L(t \sin at) = s^2 L(t \sin at)$

$2a L(\cos at) = s^2 L(t \sin at) + a^2 L(t \sin at)$

$2a L(\cos at) = (s^2 + a^2) L(t \sin at)$

$(s^2 + a^2) L(t \sin at) = 2a \dfrac{s}{a^2 + s^2}$

$= \dfrac{2as}{(s^2 + a^2)^2}$

② Find $L[t \cos at]$

Soln:- $L[t f(t)] = \dfrac{-d}{ds}[L(f(t))]$

$L[t \cos at] = \dfrac{-d}{ds}[L(\cos at)]$

$= \dfrac{-d}{ds}\left[\dfrac{s}{s^2 + a^2}\right]$

$= -\left\{\dfrac{s^2 + a^2 - s(2s)}{(s^2 + a^2)^2}\right\}$

$= -\left\{\dfrac{s^2 + a^2 - 2s^2}{(s^2 + a^2)^2}\right\} = -\left\{\dfrac{a^2 - s^2}{(s^2 + a^2)^2}\right\}$

$= \dfrac{s^2 - a^2}{(s^2 + a^2)^2}$.

(3) Find $L[te^{2t}\sin 3t]$

$$L[te^{2t}\sin 3t] = \frac{-d}{ds}\{L(e^{2t}\sin 3t)\}$$

$$= \frac{-d}{ds}\{L(\sin 3t)\}_{s\to s-2}$$

$$= \frac{-d}{ds}\left\{\left(\frac{3}{s^2+9}\right)_{s\to s-2}\right\}$$

$$= -\left\{\frac{0-3(2s)}{(s^2+9)^2}\right\}_{s\to s-2}$$

$$= \left\{\frac{6s}{(s^2+9)^2}\right\}_{s\to s-2}$$

$$= \frac{6(s-2)}{[(s-2)^2+9]^2} = \frac{6(s-2)}{(s^2-4s+4+9)^2}$$

$$= \frac{6(s-2)}{(s^2-4s+13)^2}$$

(4) Find $L\left[\frac{\sin 3t}{t}\right]$

Soln:

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds = \int_s^\infty L[f(t)]\,ds$$

$$L\left[\frac{\sin 3t}{t}\right] = \int_s^\infty L(\sin 3t)\,ds$$

$$= \int_s^\infty \left(\frac{3}{s^2+9}\right)ds = \int_s^\infty \frac{3}{s^2+3^2}\,ds$$

$$= 3\cdot\frac{1}{3}\left[\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty \qquad \left[\because \int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/3)$$

$$= \frac{\pi}{2} - \tan^{-1}(s/3)$$

$$= \cot^{-1}(s/3)$$

(5) Find $L[t^2 e^{-2t}\cos t]$

$$L[t^2 e^{-2t}\cos t] = (-1)^2 \frac{d^2}{ds^2}\{L(e^{-2t}\cos t)\}$$

$$= \frac{d^2}{ds^2}\{L(\cos t)_{s\to s+2}\}$$

$$= \frac{d^2}{ds^2}\left\{\frac{s}{s^2+1}\right\}_{s\to s+2}$$

$$= \frac{d}{ds}\left\{\frac{s^2+1-s(2s)}{(s^2+1)^2}\right\}_{s\to s+2}$$

$$= \frac{d}{ds}\left\{\frac{1-s^2}{(s^2+1)^2}\right\}_{s\to s+2}$$

$$= \left\{\frac{(s^2+1)^2(-2s)-(1-s^2)\,2(s^2+1)(2s)}{(s^2+1)^4}\right\}_{s\to s+2}$$

$$= \left\{\frac{(s^2+1)(-2s)-4s(1-s^2)}{(s^2+1)^3}\right\}_{s\to s+2}$$

$$= \frac{(s^2+4s+5)(-2s-4)+(4s+8)(s^2+4s+3)}{[(s+2)^2+1]^3}$$

$$= \frac{2s^3+12s^2+18s+4}{(s^2+4s+5)^3}$$

Integral of Laplace Transform (or) Laplace Transform of $f(t)$

If $L[f(t)] = F(s)$ and if $\underset{t\to 0}{Lt} \frac{f(t)}{t}$ exist then

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$$

Proof: $L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

Integrating w.r.t. s from s to ∞, we get,

$$\int_s^\infty F(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\,dt\right] ds$$

$$= \int_0^\infty \left[\int_s^\infty e^{-st} f(t)\,ds\right] dt$$

$$= \int_0^\infty f(t) \left[\int_s^\infty e^{-st}\,ds\right] dt = \int_0^\infty f(t) \left[\frac{e^{-st}}{-t}\right]_s^\infty dt$$

$$= \int_0^\infty f(t) \left[0 - \frac{e^{-st}}{-t}\right] dt = \int_0^\infty e^{-st} \frac{f(t)}{t}\,dt$$

$$= L\left[\frac{f(t)}{t}\right]$$

$$\therefore \quad L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds.$$

Problems:

① Find $L\left(\frac{1-\cos t}{t}\right)$

Soln: $L\left(\frac{1-\cos t}{t}\right) = \int_s^\infty L[1-\cos t]\,ds$

$$= \int_s^\infty \{L(1) - L(\cos t)\}\,ds$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right] ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \left[\log s - \log(s^2+1)^{1/2}\right]_s^\infty$$

$$= \left[\log \frac{s}{(s^2+1)^{1/2}}\right]_s^\infty = \left(\log \frac{s}{s\sqrt{1+1/s^2}}\right)_s^\infty$$

$$= \left(\log \frac{1}{\sqrt{1+1/s^2}}\right)_s^\infty = \log 1 - \log\left(\frac{1}{\sqrt{1+1/s^2}}\right)$$

$$= 0 - \log \frac{s}{\sqrt{s^2+1}}$$

$$= \log\left[\frac{s}{\sqrt{s^2+1}}\right]^{-1} = \log\left[\frac{\sqrt{s^2+1}}{s}\right]$$

② Find $L\left(\frac{e^{-3t} - e^{-4t}}{t}\right)$

Soln: $L(e^{-3t} - e^{-4t}) = \frac{1}{s+3} - \frac{1}{s+4}$

$$L\left[\frac{e^{-3t} - e^{-4t}}{t}\right] = \int_s^\infty \left(\frac{1}{s+3} - \frac{1}{s+4}\right) ds$$

$$= \cancel{\int_s^\infty \left(\frac{1}{s+3}\right.} = \left[\log(s+3) - \log(s+4)\right]_s^\infty$$

$$= \left[\log\left(\frac{s+3}{s+4}\right)\right]_s^\infty = \log\left(\frac{s+4}{s+3}\right)$$

(3) Find $L\left[\frac{1-\cos at}{t}\right]$

Soln:

$$L\left[\frac{1-\cos at}{t}\right] = \int_s^\infty L(1-\cos at)\,ds$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+a^2}\right]ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+a^2)\right]_s^\infty$$

$$= \left[\log\frac{s}{\sqrt{s^2+a^2}}\right]_s^\infty$$

$$= 0 - \log\left(\frac{s}{\sqrt{s^2+a^2}}\right)$$

$$= \log\left(\frac{\sqrt{s^2+a^2}}{s}\right)$$

(4) Find $L\left[\frac{\cos at - \cos bt}{t}\right]$

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty L[\cos at - \cos bt]\,ds$$

$$= \int_s^\infty \left[\frac{a}{a^2+s^2} - \frac{b}{s^2+b^2}\right]ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

⑤ Find the laplace transform of $e^{-t}\int_0^t t\cos t\, dt$

$$L\left[e^{-t}\int_0^t t\cos t\, dt\right] = \left[L\left(\int_0^t t\cos t\, dt\right)\right]_{s\to s+1}$$

$$\left(\because L\int_0^t f(t)dt = \frac{1}{s}L[f(s)]\right)$$

$$= \left[\frac{1}{s} L(t\cos t)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\left(-\frac{d}{ds} L(\cos t)\right)\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\left(\frac{s^2+1-2s^2}{(s^2+1)^2}\right)\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\left(\frac{1-s^2}{(s^2+1)^2}\right)\right]_{s\to s+1}$$

$$= \left[\frac{s^2-1}{s(s^2+1)^2}\right]_{s\to s+1}$$

$$= \frac{(s+1)^2-1}{(s+1)((s+1)^2+1)^2} = \frac{s^2+2s+1-1}{(s+1)[s^2+2s+1+1]^2}$$

$$= \frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$$

(6) Evaluate using Laplace Transform $\int_0^\infty t e^{-2t} \sin 3t \, dt$

$$\int_0^\infty t e^{-2t} \sin 3t \, dt = \left[\int_0^\infty e^{-st} (t \sin 3t) \, dt \right]_{s=2}$$

$$= \left[L(t \sin 3t) \right]_{s=2}$$

$$= \left[\frac{-d}{ds} L(\sin 3t) \right]_{s=2}$$

$$= \left[\frac{-d}{ds} \left(\frac{3}{s^2+9} \right) \right]_{s=2}$$

$$= \left[\frac{3(2s)}{(s^2+9)^2} \right]_{s=2}$$

$$= \left[\frac{6s}{(s^2+9)^2} \right]_{s=2}$$

$$= \frac{6(2)}{((2)^2+9)^2} = \frac{12}{169}$$