Initial Value Theorem:

If the Laplace Transform of $f(t)$ and $f'(t)$ exist and $L[f(t)] = F(s)$ then $\underset{t\to 0}{Lt}[f(t)] = \underset{s\to\infty}{Lt}[sF(s)]$

Proof:-

WKT, $L[f'(t)] = sL[f(t)] - f(0)$

$$= sF(s) - f(0)$$

$$\Rightarrow sF(s) = L[f'(t)] + f(0)$$

$$sF(s) = \int_0^\infty e^{-st} f'(t)\,dt + f(0)$$

Taking limit as $s\to\infty$ on both sides we get,

$$\underset{s\to\infty}{Lt}\, sF(s) = \underset{s\to\infty}{Lt}\left\{\int_0^\infty e^{-st} f'(t)\,dt + f(0)\right\}$$

$$= \underset{s\to\infty}{Lt}\int_0^\infty e^{-st} f'(t)\,dt + f(0)$$

$$= \int_0^\infty \underset{s\to\infty}{Lt}\, e^{-st} f'(t)\,dt + f(0)$$

$$= 0 + f(0)$$

$$= \underset{t\to 0}{Lt}\, f(t)$$

Hence $\underset{t\to 0}{Lt}\, f(t) = \underset{s\to\infty}{Lt}\, sF(s)$

Final Value Theorem:

If the Laplace Transform of $f(t)$ and $f'(t)$ exist and $L[f(t)] = F(s)$ then $\underset{t\to\infty}{Lt}[f(t)] = \underset{s\to 0}{Lt}[sF(s)]$

Proof:

WKT, $L[f'(t)] = sL[f(t)] - f(0)$

$= sF(s) - f(0)$

$\Rightarrow sF(s) = L[f'(t)] + f(0)$

Taking limit $s\to 0$ on both sides, we get

$$\underset{s\to 0}{Lt}[sF(s)] = \underset{s\to 0}{Lt}\left\{\int_0^\infty e^{-st} f'(t)\,dt + f(0)\right\}$$

$$= \int_0^\infty \underset{s\to 0}{Lt}\, e^{-st} f'(t)\,dt + f(0)$$

$$= \int_0^\infty f'(t)\,dt + f(0)$$

$$= [f(t)]_0^\infty + f(0)$$

$$= f(\infty) - f(0) + f(0) = \underset{t\to\infty}{Lt} f(t)$$

Hence $\underset{t\to\infty}{Lt} f(t) = \underset{s\to 0}{Lt}[sF(s)]$

(1) Verify the initial and final value theorem for $f(t) = 1 + e^{-t}(\sin t + \cos t)$

Soln: $F(s) = L[1 + e^{-t}\sin t + e^{-t}\cos t]$

$$= L(1) + L(\sin t)_{s\to s+1} + L(\cos t)_{s\to s+1}$$

$$= \frac{1}{s} + \left(\frac{1}{s^2+1}\right)_{s\to s+1} + \left(\frac{s}{s^2+1}\right)_{s\to s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

$$= \frac{1}{s} + \frac{s+2}{s^2+2s+2}$$

$$sF(s) = s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$$

Initial Value theorem:- $\underset{t\to 0}{Lt}\, f(t) = \underset{s\to\infty}{Lt}\, sF(s)$

$$\underset{t\to 0}{Lt}\, f(t) = \underset{t\to 0}{Lt}\left[1+e^{-t}(\sin t+\cos t)\right] = 1+1 = 2$$

$$\underset{s\to\infty}{Lt}\, sF(s) = \underset{s\to\infty}{Lt}\, s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$$

$$= \underset{s\to\infty}{Lt}\left[1 + \frac{s^2+2s}{s^2+2s\left(1+\frac{2}{s^2+2s}\right)}\right]$$

$$= \underset{s\to\infty}{Lt}\, 1 + \frac{s^2\left[1+2/s\right]}{s^2\left[1+\frac{2}{s}\left(1+\frac{2}{s^2+2s}\right)\right]}$$

$$= \underset{s\to\infty}{Lt}\, 1 + \frac{\left[1+2/s\right]}{1+2/s\left(1+2/s^2+2s\right)}$$

$$= 1+1$$

$$= 2$$

Hence $\underset{t\to 0}{Lt}\, f(t) = \underset{s\to\infty}{Lt}\, sF(s) = 2$

$\therefore$ Initial value theorem is verified

Final Value theorem: $\underset{t\to\infty}{Lt}\, f(t) = \underset{s\to 0}{Lt}\, sF(s)$

$$\underset{t\to\infty}{Lt}\ f(t) = \underset{t\to\infty}{Lt}\ [1 + e^{-t}(\sin t + \cos t)] = 1$$

$$\underset{s\to 0}{Lt}\ sF(s) = \underset{s\to 0}{Lt}\ s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$$

$$= \underset{s\to 0}{Lt}\left[1 + \frac{s^2+2s}{s^2+2s+2}\right] = 1$$

Hence $\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ sF(s) = 1$, Hence Final value theorem is verified.

Laplace Transform of Some Special functions :

Unit Step function:-

The unit step function also called Heaviside unit function is defined as,

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

This is the unit step functions at $t=a$. It can also be denoted by $H(t-a)$ or $u_a(t)$

Result:- Laplace Transform of unit step function is $\frac{e^{-as}}{s}$

(ie) $L[u(t-a)] = \frac{e^{-as}}{s}$

Proof:- $L[u(t-a)] = \int_0^\infty e^{-st}\, u(t-a)\, dt$

$$= \int_0^a e^{-st} u(t-a)\, dt + \int_a^\infty e^{-st} u(t-a)\, dt$$

$$= 0 + \int_a^\infty e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s} \quad (s>0)$$