Inverse Laplace Transform

If the Laplace Transform of $f(t)$ is $F(s)$ i.e) $L[f(t)] = F(s)$ then $f(t)$ is called an inverse Laplace transform of $F(s)$ and is written as $f(t) = L^{-1}[F(s)]$ where L^{-1} is called the inverse Laplace transform operator.

Table of Inverse Laplace Transform

$L[f(t)] = F(s)$	$L^{-1}[F(s)] = f(t)$
① $L(1) = 1/s$	$L^{-1}(1/s) = 1$
② $L(t) = 1/s^2$	$L^{-1}(1/s^2) = t$
③ $L(t^n) = \frac{n!}{s^{n+1}}$	$L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$
④ $L(e^{at}) = \frac{1}{s-a}$	$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
⑤ $L(e^{-at}) = \frac{1}{s+a}$	$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$
⑥ $L(\sin at) = \frac{a}{s^2+a^2}$	$L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$
⑦ $L\left(\frac{\sin at}{a}\right) = \frac{1}{s^2+a^2}$	$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a}$
⑧ $L(\cos at) = \frac{s}{s^2+a^2}$	$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

④ Change of Scale Property

$$L^{-1}[F(ks)] = \frac{1}{k} f\left(\frac{t}{k}\right)$$

⑤ Multiplication by S!

If $L^{-1}[F(s)] = f(t)$ and $f(0)=0$

then $L^{-1}[SF(s)] = \frac{d}{dt} L^{-1}[F(s)]$

Note!

If $f(0) \neq 0$, then $L^{-1}[SF(s)] = \frac{d}{ds} L^{-1}[F(s)] + f(0)\delta(t)$

Problem Identification:-

If $L^{-1}\left[\frac{s}{\text{quadratic eqn}}\right]$ then use result 5

If $L^{-1}\left[\frac{s}{\text{Linear eqn}}\right]$ then use the above note

⑥ Division by S:

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\,dt$$

⑦ Inverse Laplace transform of derivatives:

If $L^{-1}[F(s)] = f(t)$ then $L^{-1}[F'(s)] = -t\,L^{-1}[F(s)]$

Problem Identification:-

If $L^{-1}\left[\frac{s + \text{any term}}{(\text{quadratic eqn})^2}\right]$ then we use the above result.

(8) Note:-
If $L^{-1}[F(s)] = f(t)$ then $L^{-1}[F(s)] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds} F(s)\right]$

Problem Identification:
If L^{-1}[log function or cot fn or tan fn] then we use the above result.

Problems:-
1. Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$

Soln:-

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s^2+a^2-a^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+b^2)} - \frac{a^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{1}{s^2+b^2}\right] - a^2 L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= \frac{1}{b} L^{-1}\left[\frac{b}{s^2+b^2}\right] - \frac{a^2}{b^2-a^2} L^{-1}\left(\frac{b^2-a^2}{(s^2+a^2)(s^2+b^2)}\right)$$

$$= \frac{1}{b}\sin bt - \frac{a^2}{b^2-a^2} L^{-1}\left[\frac{1}{s^2+a^2} - \frac{1}{s^2+b^2}\right]$$

$$= \frac{1}{b}\sin bt - \frac{a^2}{b^2-a^2} \left[\frac{1}{a}\sin at - \frac{1}{b}\sin bt\right]$$

(2) Find $L^{-1}\left[\frac{2s-5}{9s^2-25}\right]$

$$L^{-1}\left[\frac{2s-5}{9s^2-25}\right] = L^{-1}\left[\frac{2s}{9s^2-25} - \frac{5}{9s^2-25}\right]$$

$$= L^{-1}\left[\frac{2s}{9(s^2-\frac{25}{9})} - \frac{5}{9(s^2-25/9)}\right]$$

$$= L^{-1}\left[\frac{2s}{9(s^2-(\frac{5}{3})^2)} - \frac{5}{9(s^2-(5/3)^2)}\right]$$

$$= \frac{2}{9} L^{-1}\left[\frac{s}{s^2-(5/3)^2}\right] - \frac{1}{3} L^{-1}\left[\frac{5/3}{s^2-(5/3)^2}\right]$$

$$= \frac{2}{9} \cosh\left(\frac{5}{3}\right)t - \frac{1}{3} \sinh\left(\frac{5}{3}\right)t$$

(3) Find $L^{-1}\left[\frac{s}{(s+2)^2+4}\right]$

$$L^{-1}\left[\frac{s}{(s+2)^2+4}\right] = \frac{d}{dt}\left[L^{-1}\left(\frac{1}{(s+2)^2+4}\right)\right]$$

$$= \frac{d}{dt}\left[e^{-2t} L^{-1}\left(\frac{1}{s^2+2^2}\right)\right]$$

$$= \frac{d}{dt}\left[\frac{e^{-2t}}{2} L^{-1}\left(\frac{2}{s^2+2^2}\right)\right]$$

$$= \frac{d}{dt}\left(\frac{e^{-2t}}{2}\sin 2t\right)$$

$$= \frac{1}{2}\frac{d}{dt}(e^{-2t}\sin 2t)$$

$$= \frac{1}{2}\left[e^{-2t}\cos 2t(2) + \sin 2t(-2)e^{-2t}\right]$$

$$= e^{-2t}\cos 2t - e^{-2t}\sin 2t$$

$$= e^{-2t}[\cos 2t - \sin 2t]$$

(4) Find $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$

soln:- $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\left(\log\frac{s+1}{s-1}\right)\right]$

$$= \frac{-1}{t} L^{-1}\left[\frac{d}{ds}(\log(s+1) - \log(s-1))\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$= \frac{-1}{t}\left(e^{-t} - e^{t}\right) = \frac{e^{t} - e^{-t}}{t}$$

$$= \frac{2}{t}\left(\frac{e^{t} - e^{-t}}{2}\right)$$

$$= \frac{2}{t}\sinh t$$

Partial Fractions:

1. Find $L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$

Soln: $\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \rightarrow (1)$

$$1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$$

Put $s=-1$

$$1 = B(-1)(-1+2)$$

$$-B = 1 \quad \boxed{B=-1}$$

Put $s=-2$

$$1 = C(-2)(-2+1)$$

$$1 = 2C \quad \boxed{C = 1/2}$$

Put $s=0$

$$1 = A(0+2)(0+1) \Rightarrow 2A = 1$$

$$\boxed{A = 1/2}$$

Sub in (1)

$$\frac{1}{s(s+1)(s+2)} = \frac{1}{2s} + \frac{(-1)}{s+1} + \frac{1}{2(s+2)}$$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}$$

$$= \frac{1}{2}\left[1 - 2e^{-t} + e^{-2t}\right]$$

② Find $L^{-1}\left[\frac{s^2}{(s+1)(s^2+4)}\right]$

Soln :-

$$\frac{s^2}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

$$s^2 = A(s^2+4) + (Bs+C)(s+1)$$

Put $s = -1$

$$(-1)^2 = A((-1)^2 + 4) + (B(-1)+C)(-1+1)$$

$$1 = A(5)$$

$$\boxed{A = 1/5}$$

Put $s = 0$

$$0 = A(0+4) + (B(0)+C)(0+1)$$

$$0 = 4A + C$$

$$C = -4A$$

$$\boxed{C = -4/5}$$

Put $s = -4$

$$16 = A(16+4) + (B(-4)+C)(-4+1)$$

$$16 = 20A + (-4B+C)(-3)$$

$$16 = 20(1/5) + 12B - 3(-4/5)$$

$$16 = 4 + 12B + 12/5$$

$$12B = 16 - 4 - 12/5$$

$$12B = 12 - 12/5$$

$$B = 1 - 1/5$$

$$\boxed{B = 4/5}$$

$$L^{-1}\left[\frac{s^2}{(s+1)(s^2+4)}\right] = L^{-1}\left[\frac{1/5}{s+1} + \frac{(4/5)(s - 4/5)}{s^2+4}\right]$$

$$= \frac{1}{5} L^{-1}\left[\frac{1}{s+1}\right] + \frac{4}{5} L^{-1}\left(\frac{s}{s^2+4}\right) - \frac{4}{5}\left(\frac{1}{s^2+4}\right)$$

$$= \frac{1}{5} e^{-t} + \frac{4}{5}\cos 2t - \frac{4}{5}\frac{\sin 2t}{2}$$

$$= \frac{1}{5} e^{-t} + \frac{4}{5}\cos 2t - \frac{4}{10}\sin 2t$$

(3) Find $L^{-1}\left[\frac{s^2+2s+1}{(s+3)(s-3)(s+1)}\right]$ Ans: $\frac{1}{3}e^{-3t} + \frac{2}{3}e^{3t}$

(4) Find $L^{-1}\left[\frac{s}{(s-3)(s^2+4)}\right]$ Ans: $\frac{3}{13}e^{3t} - \frac{3}{13}\cos 2t + \frac{2}{13}\sin 2t$

Convolution :-

If $f(t)$ and $g(t)$ are two functions defined for $t \geq 0$ then the convolution of $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = (f * g)(t) = \int_0^t f(u)\, g(t-u)\, du$$

Note :- $f(t) * g(t) = g(t) * f(t)$

Convolution Theorem :-

If $f(t)$ and $g(t)$ are two Laplace transformable functions defined for $t \geq 0$ then $L[f(t) * g(t)]$ is given by,

$$L[f(t) * g(t)] = L[f(t)] \cdot L[g(t)]$$

$$L^{-1}[F(s) \cdot G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$$

Problems :-

(1) Using convolution theorem, find the inverse transform of

i) $\dfrac{s^2}{(s^2+a^2)^2}$ (ii) $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$ (iii) $\dfrac{1}{(s+a)(s+b)}$

Soln :-

i) $$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)} \cdot \frac{s}{(s^2+a^2)}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+a^2}\right]$$

$$= \cos at * \cos at$$

$$= \int_0^t \cos au \, \cos a(t-u)\, du$$

$$= \int_0^t \left[\frac{\cos(au+at-au)+\cos(au-at+au)}{2}\right] du$$

$\cos A \cos B = \frac{\cos(A+B)+\cos(A-B)}{2}$

$$= \frac{1}{2}\int_0^t \left[\cos at + \cos(2au-at)\right] du$$

$$= \frac{1}{2}\left[\cos at \cdot u + \frac{\sin(2au-at)}{2a}\right]_0^t$$

$$= \frac{1}{2}\left[\cos at \cdot t + \frac{\sin(2at-at)}{2a} - \frac{\sin(0-at)}{2a}\right]$$

$\sin(-\theta) = -\sin\theta$

$$= \frac{1}{2}\left[t\cos at + \frac{\sin at}{2a} + \frac{\sin at}{2a}\right]$$

$$= \frac{1}{2}\left[t\cos at + \frac{2\sin at}{2a}\right] = \frac{1}{2}\left[t\cos at + \frac{\sin at}{a}\right]$$

$$= \frac{1}{2a}\left[at\cos at + \sin at\right]$$

(ii) $L^{-1}\left[\frac{s}{(s^2+a^2)(s^2+b^2)}\right]$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+b^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{b}\cdot\frac{b}{s^2+b^2}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * \frac{1}{b}L^{-1}\left[\frac{b}{s^2+b^2}\right] = \cos at * \frac{1}{b}\sin bt$$

$$= \frac{1}{b}\int_0^t \cos au \, \sin b(t-u)\, du.$$

$\sin(A+B)+\sin(A-B) = 2\cos A\sin B$

$$= \frac{1}{b}\int_0^t \frac{\sin(bt-bu+au)+\sin(bt-bu-au)}{2}\,du.$$

$$\because \sin A\cos B = \frac{\sin(A+B)+\sin(A-B)}{2}$$

$$= \frac{1}{2b}\int_0^t \sin\left[bt+(a-b)u\right]+\sin\left[bt-(a+b)u\right]du$$

$$= \frac{1}{2b}\left[\frac{-\cos(bt+(a-b)u)}{a-b} - \frac{\cos(bt-(a+b)u)}{-(a+b)}\right]_0^t$$

$$= \frac{1}{2b}\left[\frac{\cos(bt-(a+b)u)}{a+b} - \frac{\cos(bt+(a-b)u)}{a-b}\right]_0^t$$

$$= \frac{1}{2b}\left\{\left[\frac{\cos(\cancel{bt}-at-\cancel{bt})}{a+b} - \frac{\cos(\cancel{bt}+at-\cancel{bt})}{a-b}\right] - \left[\frac{\cos bt}{a+b} - \frac{\cos bt}{a-b}\right]\right\}$$

$\cos(-\theta)=\cos\theta$

$\sin(-\theta)=-\sin\theta$

$$= \frac{1}{2b}\left\{+\frac{\cos at}{a+b} - \frac{\cos at}{a-b} - \frac{\cos bt}{a+b} + \frac{\cos bt}{a-b}\right\}$$

$$= \frac{1}{2b}\left\{\frac{1}{a+b}\left[\cos at - \cos bt\right] - \frac{1}{a-b}\left[\cos at - \cos bt\right]\right\}$$

$$= \frac{\cos at - \cos bt}{2b(a^2-b^2)}\left\{\frac{a-b-a-b}{a^2-b^2}\right\} = \frac{(\cos at - \cos bt)(-2b)}{2b(a^2-b^2)}$$

$$= \frac{\cos bt - \cos at}{a^2-b^2}.$$

(iii) $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$

$$L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = L^{-1}\left[\frac{1}{s+a}\cdot\frac{1}{s+b}\right]$$

$$= L^{-1}\left[\frac{1}{s+a}\right]\cdot L^{-1}\left[\frac{1}{s+b}\right]$$

$$= e^{-at} * e^{-bt}$$

$$= \int_0^t e^{-au}\cdot e^{-b(t-u)}\,du$$

$$= \int_0^t e^{-au-bt+bu}\,du$$

$$= \int_0^t e^{-bt-(a-b)u}\,du.$$

$$= \left[\frac{e^{-bt-(a-b)u}}{-(a-b)}\right]_0^t$$

$$= \left[\frac{e^{-bt-at+bt}}{-(a-b)} - \frac{e^{-bt}}{-(a-b)}\right]$$

$$= -\left[\frac{e^{-at}}{a-b} + \frac{e^{-bt}}{a-b}\right]$$

$$= -\left[\frac{e^{-at}+e^{-bt}}{a-b}\right]$$