Application of Laplace transforms to Differential equations:-

If $L[f(t)] = F(s)$ then

$L[y'(t)] = sL(y) - y(0)$

$L[y''(t)] = s^2L(y) - sy(0) - y'(0)$

① Solve the differential equations using Laplace Transform $y''+4y'+4y = e^{-t}$ given that $y(0)=0$, and $y'(0)=0$.

$y''+4y'+4y = e^{-t}$

Taking Laplace Transform on both sides,

$L(y''+4y'+4y) = L(e^{-t})$

$L(y'') + 4L(y') + 4L(y) = \frac{1}{s+1}$

$[s^2L(y) - sy(0) - y'(0)] + 4[sL(y) - y(0)] + 4L(y) = \frac{1}{s+1}$

Given: $y(0)=0$, $y'(0)=0$.

$[s^2L(y) - s(0) - 0] + 4[sL(y) - (0)] + 4L(y) = \frac{1}{s+1}$

$s^2L(y) + 4sL(y) + 4L(y) = \frac{1}{s+1}$

$(s^2+4s+4)L(y) = \frac{1}{s+1}$

$L(y)(s+2)^2 = \frac{1}{s+1}$

$L(y) = \frac{1}{(s+1)(s+2)^2}$

$$y = L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right]$$

$$\frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$1 = A(s+2)^2 + B(s+2)(s+1) + C(s+1)$$

Put $s = -2 \Rightarrow 1 = A(0) + B(0) + C(-2+1)$

$$\boxed{C = -1}$$

Put $s = -1 \Rightarrow 1 = A(-1+2)^2$

$$\boxed{A = 1}$$

Put $s = 0 \Rightarrow 1 = A(2)^2 + B(2)(1) + C(1)$

$$1 = 4A + 2B + C$$

$$1 = 4(1) + 2B + (-1)$$

$$1 = 2B + 3$$

$$2B = -2 \quad \boxed{B = -1}$$

$$\frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

$$y = L^{-1}\left(\frac{1}{(s+1)(s+2)^2}\right)$$

$$= L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s+2}\right) - L^{-1}\left(\frac{1}{(s+2)^2}\right)$$

$$= e^{-t} - e^{-2t} - te^{-2t}$$