Simultaneous First order Linear Equations with Constant Coefficients

Let x, y be the two dependent variables and t be the independent variable, the equation involving their derivatives are called Simultaneous differential equations. The number of equations is the same as the number of dependent variables

1. Solve $\frac{dx}{dt} + y = e^{t}$, $\frac{dy}{dt} - x = e^{-t}$

Solution :-

$$\frac{dx}{dt} + y = e^{t}$$

$$\frac{dy}{dt} - x = e^{-t}$$

using the operator $D = \frac{d}{dt}$

$$Dx + y = e^{t} \rightarrow (1)$$

$$Dy - x = e^{-t} \rightarrow (2).$$

Here we can eliminate either x or y

Eliminate x by solving (1) & (2)

$$(1) + (2)D \Rightarrow D^{2}y + y = D(e^{-t}) + e^{t}$$

$(D^2+1)y = e^t - e^{-t}$

Auxillary equation is $m^2+1=0$

$\therefore m = \pm i$

$\therefore$ Complementary function, $CF = A\cos t + B\sin t$

$PI_1 = \frac{e^t}{D^2+1} = \frac{e^t}{2}$

$PI_2 = \frac{e^{-t}}{D^2+1} = \frac{e^{-t}}{2}$

$y = CF + PI$

$= A\cos t + B\sin t + \frac{1}{2}[-e^{-t} + e^t]$

$y = A\cos t + B\sin t + \sinh t$

Sub in (2) $\quad Dy - x = e^{-t}$

$x = Dy - e^{-t}$

$= \frac{d}{dt}(A\cos t + B\sin t + \sinh t) - e^{-t}$

$x = -A\sin t + B\cos t + \cosh t - e^{-t}$

$\therefore$ The solution is $x = B\cos t - A\sin t + \cosh t - e^{-t}$

$y = A\cos t + B\sin t + \sinh t$

2) Solve $\frac{dx}{dt} + y = \sin t$

$\frac{dy}{dt} + x = \cos t$ given that $t=0, x=1,$

and $y=0$.

Solution :- $Dx + y = \sin t \rightarrow (1)$

$Dy + x = \cos t \rightarrow (2)$.

Eliminating x,

$(1) - (2)D, \Rightarrow Dx + y - D^2y - Dx = \sin t - D(\cos t)$

$-D^2y + y = \sin t + \sin t \Rightarrow -(D^2-1)y = 2\sin t$

$(D^2-1)y = -2\sin t$

AE is $m^2 - 1 = 0 \Rightarrow m^2 = 1$

$m = \pm 1$

$CF = Ae^{t} + Be^{-t}$

$PI = \frac{1}{D^2-1} - 2\sin t$

$= -2 \frac{1}{-1-1} \sin t = \frac{-2}{-2} \sin t$

$= \sin t$

$y = CF + P.I = Ae^{t} + Be^{-t} + \sin t$

Sub in ② $\Rightarrow$ $Dy + x = \cos t$

$$x = \cos t - Dy$$

$$= \cos t - \frac{d}{dt}\left(Ae^{t} + Be^{-t} + \sin t\right)$$

$$= \cos t - Ae^{t} + Be^{-t} - \cos t$$

$$x = -Ae^{t} + Be^{-t}$$

$\therefore$ $x = -Ae^{t} + Be^{-t}$ & $\rightarrow$ ③

$y = Ae^{t} + Be^{-t} + \sin t \rightarrow$ ④

Now using the conditions given, we can find A & B, Sub. $x=1, y=0$ & $t=0$ in ③&④

③ $\Rightarrow$ $1 = -Ae^{0} + Be^{-0}$

$\Rightarrow -A + B = 1 \rightarrow$ ⑤

④ $\Rightarrow$ $0 = Ae^{0} + Be^{-0} + \sin(0)$

$A + B = 0 \rightarrow$ ⑥

Solving ⑤ & ⑥

$2B = 1$

$\boxed{B = 1/2}$.

Sub in ⑤

$-A = 1 - B$

$\boxed{A = -1/2}$

$$\therefore \; x = \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} = \cosh t$$

$$y = \frac{-1}{2}e^{t} + \frac{1}{2}e^{-t} + \sin t$$

$$= \sin t - \sinh t$$

③ Solve $\frac{dx}{dt} + 2y = -\sin t$

$$\frac{dy}{dt} - 2x = -\cos t$$

④ Solve $\frac{dx}{dt} + 2x - 3y = t$

$$\frac{dy}{dt} - 3x + 2y = e^{2t}$$