Unit 3
Complex Differentiation

1. **Define analytic function of a complex variable.**
   A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighborhood of that point.

2. **Show that $u = 3x^2y - y^3$ is a harmonic function.**
   Given,
   $$u = 3x^2y - y^3$$
   $$\frac{\partial u}{\partial x} = 6xy$$
   $$\frac{\partial^2 u}{\partial x^2} = 6y$$
   $$\frac{\partial u}{\partial y} = 3x^2y - y^3$$
   $$\frac{\partial^2 u}{\partial y^2} = -6y$$
   $$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y - 6y = 0$$
   Hence u is harmonic.

3. **Define bilinear transformation or [Mobius Transformation]**
   The transformation $w = \frac{az+b}{cz+d}$, $ad - bc \neq 0$ where a, b, c, d are complex numbers is called a bilinear transformation.

4. **Verify the function $\phi(x, y) = e^x \sin y$ is harmonic or not.**
   Given $\phi = e^x \sin y$
   To prove $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$
   $$\frac{\partial \phi}{\partial x} = e^x \sin y \quad \frac{\partial \phi}{\partial x} = e^x \cos y$$
   $$\frac{\partial^2 \phi}{\partial x^2} = e^x \sin y \quad \frac{\partial^2 \phi}{\partial y^2} = -e^x \sin y$$
   $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = e^x \sin y - e^x \sin y$$
   = 0
   $\therefore$ ϕ **satisfies the Laplace equation.**
   ϕ **is harmonic.**

5. **Find the invariant of bilinear transformation** $w = \frac{1+z}{1-z}$.
   $$w = \frac{1+z}{1-z}$$

Replace w by z,

$$z = \frac{1+z}{1-z}$$

$$z - z^2 = 1 + z$$

$$1 + z - z + z^2 = 0$$

$$z^2 + 1 = 0$$

$$z = \pm i$$

The invariant points are $\pm i$.

6. **Conformal Mapping**

   A mapping $w = f(z)$ is said to be conformal at $z = z_0$ if it preserves the angle between any two curves through z_0 in z plane both in magnitude & direction.

7. **Find the critical points of the transformation $w = z + \frac{1}{z}$.**

   Let $w = z + \frac{1}{z}$

$$\Rightarrow \frac{dw}{dz} = 1 - \frac{1}{z^2} = \frac{z^2 - 1}{z^2}$$

$$\therefore \frac{dz}{dw} = \frac{z^2}{z^2 - 1}$$

   Now the critical points are $\pm 1, 0$.

8. **Obtain the invariant point of the transformation $w = 2 - \frac{2}{z}$**

   The invariant points are given by

$$z = 2 - \frac{2}{z}; z = \frac{2z - 2}{z}$$

$$z^2 = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$$\therefore z = 1 \pm i$$

9. **Find the fixed point of the transformation $w = \frac{6z - 9}{z}$**

   The fixed points are given by replacing $w = z$

$$\text{ie.,}, w = \frac{6z-9}{z}$$

$$z = \frac{6z-9}{z} \Rightarrow z^2 = 6z - 9$$

$$\Rightarrow (z-3)^2 = 0 \Rightarrow z = 3,3$$

   The fixed points are 3,3.

10. **Test the analyticity of the function $w = \sin z$**

   Let $w = f(z) = \sin z$

$$u + iv = \sin(x + iy)$$

$$u + iv = \sin x \cos iy + \cos x \sin iy$$

$$u + iv = \sin x \cosh y + i \cos x \sinh y$$

$u = \sin x \cosh y$ $\quad$ $v = \cos x \sinh y$

$u_x = \cos x \cosh y$ $\quad$ $v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$ $\quad$ $v_y = \cos x \cosh y$

$\therefore u_x = v_y$ and $u_y = -v_x$

C.R equations are satisfied.
Hence the function is analytic.

11. Determine whether the function $2xy + i(x^2 - y^2)$ is analytic or not.

Let $f(z) = 2xy + i(x^2 - y^2)$

$u = 2x, v = x^2 - y^2$

$\frac{\partial u}{\partial x} = 2y \qquad \frac{\partial v}{\partial x} = 2x$

$\frac{\partial u}{\partial y} = 2x \qquad \frac{\partial v}{\partial y} = -2y$

$u_x \neq v_y$ and $u_y \neq -v_x$

C-R equations are not satisfied.
Hence f(z) is not an analytic function.

12. Define C-R equations.

$u_x = v_y$ ....(1)

$u_y = -v_x$ ....(2)