Unit 4
Complex Integration

1. **State Cauchy's integral formula.**

   Solution:

   If f(z) is analytic inside and on a simple closed curve C and 'a' be any point inside C then $\int_c \frac{f(z)}{z-a}dz = 2\pi i.f(a)$ **where the integration being taken in the positive direction around C.**

2. **Expand log(1+z) Taylor's series about z=0.**

   Solution:

$$f(z) = \log(1+z) \Rightarrow f(0) = \log 1 = 0$$

$$f'(z) = \frac{1}{1+z} \Rightarrow f'(0) = 1$$

$$f''(z) = -\frac{1}{(1+z)^2} \Rightarrow f''(0) = -1$$

$$f'''(z) = \frac{2(1+z)}{(1+z)^4} = \frac{2}{(1+z)^3} \Rightarrow f'''(0) = 2$$

The Taylor series of f(z) about the point z=0 is given by

$$f(z) = f(0) + \frac{z}{1!}f'(0) + \frac{z^2}{2!}f''(0) + \frac{z^3}{3!}f'''(0) + ...$$

$$= 0 + \frac{z}{1!} - \frac{z^2}{2!} + 2\frac{z^3}{3!} - ...$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - ...$$

3. **Find Laurent's series of** $f(z) = z^2 e^{\frac{1}{z}}$ **about z=0.**

   Solution:

   Clearly f(z) is not analytic at z=0.

$$f(z) = z^2 e^{\frac{1}{z}}$$

$$= z^2\left[1 + \frac{\left(\frac{1}{z}\right)}{1!} + \frac{\left(\frac{1}{z}\right)^2}{2!} + ...\right]$$

$$= z^2 + \frac{z}{1!} + \frac{1}{2!} + ...$$

4. State Cauchy's Residue theorem.

If f(z) be analytic at all points inside and on a simple closed curve C,except for a finite number of isolated singularities $z_1, z_2, \ldots, z_n$ inside C,then

$$\int_c f(z)dz = 2\pi i[\text{sum of the residues of } f(z) \text{ at } z_1, z_2, \ldots, z_n]$$

$$= 2\pi i \sum_{i=1}^{n} R_i \text{ where } R_i \text{ is the residue of } f(z) \text{ at } z = z_i$$

5. Define pole and simple poles.

A point z=a is said to be a pole f(z) of order n if we can find a positive integer n such that $\lim_{z \to a} (z-a)^n f(z) \neq 0$

A pole of order one is called a simple pole.

$$f(z) = \frac{1}{(z-4)^2 (z-3)^3 (z-1)}$$

Example :

Hence z=1 is a simple pole of order 1

Z=4 is a simple pole of order 2

Z=3 is a simple pole of order 3.

6. Find the singular points of $f(z) = \frac{1}{\sin \frac{1}{z-a}}$ **state their nature.**

Solution:

f(z) has an finite number of poles which are given by

$$\frac{1}{z-a} = n\pi, n = \pm 1, \pm 2, \ldots.$$

$$\text{i.e } z - a = \frac{1}{n\pi}$$

$$z = a + \frac{1}{n\pi}$$

But z=a also is a singular point.

It is an essential singularity

It is a limit point of the poles

So it is a non isolated singularity.

7. Find the singularity of the function f(z)= sinz/z

Removable Singularity

8. Find the principal part and residue at the pole of

$$f(z) = \frac{2z+3}{(z+2)^2} = (2z+3)(z+2)^{-2}$$

[since principal part is negative powers]

$$(z+2)^{-2} \Rightarrow z = -2 \text{ is a singular pole of order } 2$$

$$\text{Res}\{f(z)\}_{z=-2} = \lim_{z\to -2} \frac{d}{dz}\left[(z+2)^2 \cdot \frac{(2z+3)}{(z+2)^2}\right] = 2$$

9. Find the regularities of the function $f(z) = \dfrac{\cot \pi z}{(z-a)^3}$

Given $f(z) = \dfrac{\cot \pi z}{(z-a)^3} = \dfrac{\cos \pi z}{\sin \pi z (z-a)^3}$

Singular points are poles and are given by Dr=0.

$\sin \pi z = 0, \quad (z-a)^3 = 0 \Rightarrow z = a$ is a singular pole of order 3

$\sin \pi z = \sin n \sin n\pi \quad$ where $n = 0, \pm 1, \pm 2, \ldots\ldots$

$\pi z = n\pi \Rightarrow n = \pi$

$z = n = 0, \pm 1, \pm 2, \ldots\ldots$ are simple poles.