UNIT - III

Solutions of Equations.

Newton's Method / Newton's - Raphson Method.

Formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{and } f'(x_n) \neq 0$$

1) Find the smallest positive root of the equation $x^3 - 2x + 0.5 = 0$.

Solution:

Given: $f(x) = x^3 - 2x + 0.5$

$f'(x) = 3x^2 - 2$

$f(0) = 0.5$ (+ve)

$f(1) = 1^3 - 2(1) + 0.5$

$= 1 - 2 + 0.5$

$= -0.5$ (-ve).

$\therefore$ The root lies b/w 0 and 1.

Since $|f(0)| = |f(1)|$, Let us assume $x_0 = 0$

Newton Raphson Formula.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

put $n = 0$, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ $\qquad f'(x) = 3x^2 - 2$, $f'(0) = 0 - 2 = -2$

$= 0 - \frac{0.5}{-2}$

$$\boxed{x_1 = 0.25}$$

$f(0.25) = 0.0156$

$f'(0.25) = 3(0.25)^2 - 2 = -1.8125$

put $n = 1$, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 0.25 - \frac{0.0156}{-1.8125}$

$$\boxed{x_2 = 0.2586}$$

put $n=2$, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$f(0.2586) = (0.2586)^3 - 2(0.2586) + 0.5$

$f'(0.2586) = 3(0.2586)^2 - 2$

$= 0.2586 -$

$x_3 = 0.2586$

Since x_2 and x_3 are equal roots, the smallest positive root is 0.2586.

2) Using Newton's iterative method find the root b/w 0 and 1 of $x^3 = 6x - 4$ correct to two decimal places.

Solution:

Given : $f(x) = x^3 - 6x + 4$

$f'(x) = 3x^2 - 6$

$f(0) = 0 - 0 + 4 = 4$ (+ve)

$f(1) = 1 - 6 + 4 = -1$ (-ve).

Hence the root lies b/w 0 and 1.

Since $|f(0)| > |f(1)|$, the root is nearer to 1. Assume $x_0 = 1$

Newton Raphson Formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

put $n=0$ $\quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$f(0) = -1$ / $f'(0) = 3 - 6 = 3 - 6 = -3$

$x_1 = 1 - \frac{-1}{-3}$

$= \frac{3-1}{3} = \frac{2}{3} = 0.666$.

$\boxed{x_1 = 0.666}$

put $n=1$, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$f(0.67) = (0.67)^3 - 6(0.67) + 4$

$f'(0.67) = 3(0.67)^2 - 6$

$x_2 = 0.67 - \frac{0.28}{-4.653}$

$= 0.67 + 0.060$

$\boxed{x_2 = 0.73}$

put $n=2$, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$f(0.73) = (0.73)^3 - 6(0.73) + 4$

$f'(0.73) = 3(0.73)^2 - 6$

$x_3 = 0.73 - \frac{0.009}{-4.401}$

$= 0.73 + 0.00204$

$\boxed{x_3 = 0.73}$

Since x_2 and x_3 are equal, the root is 0.73.

3) Compute the real root of $x \log_{10} x = 1.2$ correct to 3 decimal places using Newton-Raphson Method.

Solution: Given: $f(x) = x \log_{10} x - 1.2$

$f'(x) = \log_{10} x + x \cdot \frac{1}{x} \log_{10} e$

$\because \frac{d}{dx} \log_a x = \frac{1}{x} \log_a e$ (log & e are inverses)

$f'(x) = \log_{10} x + 0.4343$ [$\log_a e = 0.4343$

$f(0) = -1.2$ | $f(1) = 1 \cdot \log_{10} 1 - 1.2 = -1.2$ | $f(2) = 2 \cdot \log_{10} 2 - 1.2 = -0.59$

Newton - Raphson Formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

The first approximation to the root is given by

$n=0$

$x_1 = x_0 -$

$|f(3)| < |f(2)|$

$\therefore$ We assume $x_0 = 3$

put $n=0$ in formula, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$f(x_0) = 3\log_{10} 3 - 1.2$ | $f'(x_0) = \log_{10} 3 + 0.4343$

$= 0.9114$

$= 0.2314$

$$x_1 = 3 - \frac{0.2314}{0.9114} = 2.746$$

$$\boxed{x_1 = 2.746}$$

put $n=1$, $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$f(x_1) = f(2.746) = 2.746 \log_{10} 2.746 - 1.2$

$= 0.0047$

$f'(x_1) = f'(2.746) = \log_{10} 2.746 + 0.4343$

$= 0.8930$

$$x_2 = 2.746 - \frac{0.0047}{0.8930}$$

$$\boxed{x_2 = 2.741}$$

put $n=2$, $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$f(x_2) = f(2.741) = 2.741 \log_{10} 2.741 - 1.2$

$= 0.0003$

$f'(x_2) = f'(2.741) = \log_{10} 2.741 + 0.4343$

$= 0.8722$

$$x_3 = 2.741 - \frac{0.0003}{0.8722}$$

$$\boxed{x_3 = 2.741}$$

Since x_2 and x_3 are equal, the root correct to three decimal places is 2.741

4) Find the real root of $3x - \cos x - 1 = 0$ by N-R Method.

~~Find~~

Solution: Let $f(x) = 3x - \cos x - 1$ —— (1)

diff with respect (1) equation

$f'(x) = 3 - (-\sin x)$

$f'(x) = 3 + \sin x$ —— (2)

T & E (1)

$f(0) = 0 - \cos(0) - 1$

$f(0) = -2$ (-ve)

$f(1) = 3 - \cos(1) - 1$

$f(1) = 3 - \cos(1) - 1$

$f(1) = 1.0002$ (+ve)

Since $|f(0)| > |f(1)|$

Assume $x_0 = \frac{0+1}{2} = 0.5$

$\boxed{x_0 = 0.5}$

N-R Formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Put $n = 0$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$f(0.5) = 3(0.5) - \cos(0.5) - 1 = 0.4999$

$f'(0.5) = 3 + \sin(0.5) = 3.0087$

$$x_1 = 0.5 - \frac{(-0.4999)}{3.0087}$$

$\boxed{x_1 = 0.6662}$

Put $n=1$,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$f(0.6662) = 3(0.6662) - \cos(0.6662) - 1$

$= -0.0013$

$f'(0.6662) = 3 + \sin(0.6662) = 3.0116$

$$x_2 = 0.6662 - \frac{(-0.0013)}{3.0116} = 0.6666$$

$$\boxed{x_2 = 0.6666}$$

Put $n=2$,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$f(0.6666) = 3(0.6666) - \cos(0.6666) - 1$

$= -0.0001$

$f'(0.6666) = 3 + \sin(0.6666) = 3.0116$

$$x_3 = 0.6666 - \frac{(-0.0001)}{3.0116} = 0.6666$$

Since x_2 and x_3 are equal, the root is

$x = 0.6666$

(5) Evaluate $\sqrt{12}$ to four decimal places by Newton's - Raphson Method.

Soln: Given: Let $x = \sqrt{12}$

$\Rightarrow x^2 = 12$

$\Rightarrow x^2 - 12 = 0$.

$\Rightarrow$ Let $f(x) = x^2 - 12$

$f'(x) = 2x$

T&E

$f(0) = -12$

$f(1) = -11$

$f(2) = -8$

$f(3) = -3$ (-ve)

$f(4) = 4$ (+ve)

$|f(3)| < |f(4)|$

$\therefore$ Assume $x_0 = 3$

$n=0,\quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{f(3)}{f'(3)}$

$f(3) = 3^2 - 12 = 9-12 = -3$

$f'(3) = 2(3) = 6$

$x_1 = 3 - \frac{(-3)}{6} = 3 + \frac{1}{2} = 3.5$

$\boxed{x_1 = 3.5}$

$n=1,\quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.5 - \frac{f(3.5)}{f'(3.5)}$

$f(3.5) = (3.5)^2 - 12 = 0.25$

$f'(3.5) = 2(3.5) = 7.0$

$x_2 = 3.5 - \frac{0.25}{f(7.0)}$

$\boxed{x_2 = 3.4643}$

$n=2,\quad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$= 3.4643 - \frac{f(3.4643)}{f'(3.4643)}$

$f(3.4643) = (3.4643)^2 - 12 = 0.0014$

$f'(3.4643) = 2(3.4643)$

$= 6.9286$

$x_3 = 3.4643 - \frac{0.0014}{6.9286}$

$\boxed{x_3 = 3.4641}$

$n=3,\quad x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$

$= 3.4641 - \frac{f(3.4641)}{f'(3.4641)}$

$f(3.4641) = (3.4641)^2 - 12$

$= -0.00001$

$f'(3.4641) = 6.9282$

$x_3 = 3.464 - \frac{(-0.00001)}{6.9282}$

$\boxed{x_3 = 3.4641}$

Since x_2 and x_3 are equal, root is $x = 3.4641$ //

(6) Find the pth root of positive number N and hence find the cube root of 17, using N-R Method.

Solution

$x = N^{1/p} \qquad x^p = N$

$f(x) = x^p - N$

$f'(x) = p x^{p-1}$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$= x_n - \frac{(x_n^p - N)}{p x_n^{p-1}}$

$= \frac{p x_n^p - x_n^p + N}{p x_n^{p-1}}$

$\boxed{x_{n+1} = \frac{x_n^p (p-1) + N}{p x_n^{p-1}}}$

is the N-R formula for pth root of +ve no. N

$N = 17;\ x_0 = 2;\ p = 3$

cube root of 17

$x = (17)^{1/3} \Rightarrow x^3 - 17 = 0$

$f(0) = -17$

$f(1) = -16$

$f(2) = -9$ (-ve)

$f(3) = 10$ (+ve)

$\boxed{x_0 = 2}$

$x_1 = 2.75$

$x_2 = 2.58264$

$x_3 = 2.57133$

$x_4 = 2.57128$

$x_5 = 2.57128$

cube root is 2.57128 //

N-R Method

Find the N-R formula to find the value of $1/N$ where N is a real no, hence evaluate $\frac{1}{26}$ correct to 4 decimal places.

Solution:

Given $x = \frac{1}{N}$

$$\frac{1}{x} = N.$$

$$\frac{1}{x} - N = 0.$$

Let $f(x) = \frac{1}{x} - N.$

$$f'(x) = -\frac{1}{x^2}.$$

N-R

$$x_{n+1} = x_n - \frac{\left(\frac{1}{x_n} - N\right)}{\left(-\frac{1}{x_n^2}\right)}$$

$$= x_n + x_n^2\left(\frac{1}{x_n} - N\right)$$

$$= \cancel{x_n^2} + \cancel{x_n^2} - \cancel{x_n^2 N}$$

$$= \cancel{2x_n^2 - x_n^2 N}$$

$$= x_n + x_n - x_n^2 N = 2x_n - x_n^2 N$$

$$\boxed{x_{n+1} = x_n\left[2 - Nx_n\right]}$$ is the N-R formula to find $\frac{1}{N}$,

$1/26$

$N = 26, \quad n = 0$

$$x_1 = x_0\left[2 - 26x_0\right]$$

$f(0) = -26$

$f(1) = \frac{1}{1} - 26 = -25$ (-ve)

$f(2) = \frac{1}{2} - 26 = -25.5$ (-ve)

since it takes long process.

Let us take $x_0 = \frac{1}{25} = 0.04$, nearer to the given N.

$x_0 = 0.04$

$n = 0$

$$x_1 = x_0\left[2 - 26x_0\right] =$$

$$= 0.04\left[2 - 26(0.04)\right) = 0.0384$$

$x_2 = 0.0384$ Ans: 0.0384