Numerical Differentiation:

Forward Difference formula.

$$f'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$f''(x_0) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right]$$

Backward Difference formula

$$f'(x_0) = \frac{1}{h}\left[\nabla y_0 + \frac{1}{2}\nabla^2 y_0 + \frac{1}{3}\nabla^3 y_0 + \frac{1}{4}\nabla^4 y_0 + \cdots\right]$$

$$f''(x_0) = \frac{1}{h^2}\left[\nabla^2 y_0 + \nabla^3 y_0 + \frac{11}{12}\nabla^4 y_0 + \cdots\right]$$

Problem 1.: Coa-A

Find the first, second derivatives of the function tabulated below at the point $x = 1.5$ and $x = 4.0$

x	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$	3.375	7.0	13.625	24.0	38.875	59.0

Solution:

To find at $x = 1.5$ → forward formula.

at $x = 4.0$ → backward formula.

Given: $x_0 = 1.5$

$h = 2.0 - 1.5 = 0.5$

Newton's forward & backward formula to find derivatives. (4) formulae.

The difference table.

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.5 (x_0)	3.375 (y_0)	3.625 (Δy_0)			
2.0	7.0		3.0 ($\Delta^2 y_0$)		
		6.625		0.75 ($\Delta^3 y_0$)	
2.5	13.625		3.75		0 ($\Delta^4 y_0$)
		10.375		0.75	
3.0	24.0		4.50		0 ($\nabla^4 y_0$)
		14.875		0.75	
3.5	38.875		5.25	($\nabla^3 y_0$)	
		20.125 (∇y_0)	($\nabla^2 y_0$)		
4.0 (x_n)	59.0 (y_n)				

Forward

$$f'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$f'(1.5) = \frac{1}{0.5}\left[3.625 - \frac{1}{2}(0.75) + \frac{1}{3}(0.75) + 0\right]$$

$$\boxed{f'(1.5) = 4.75}$$

$$f''(x_0) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right]$$

$$f''(1.5) = \frac{1}{(0.5)^2}\left[3.0 - 0.75 + 0\right]$$

$$\boxed{f''(1.5) = 9.0}$$

Backward

$$f'(x_0) = \frac{1}{h}\left[\nabla y_0 + \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\nabla^3 y_0 + \cdots\right]$$

$$f'(4.0) = \frac{1}{0.5}\left[20.125 + \frac{1}{2}(2.25) + \frac{1}{3}(0.75)\right]$$

$$= \frac{1}{0.5}\left[20.125 + 2.625 + 0.25\right]$$

$$\boxed{f'(4.0) = 46}$$

$$f''(x_0) = \frac{1}{h^2}\left[\nabla^2 y_0 + \nabla^3 y_0 + \frac{11}{12}\nabla^4 y_0 + \cdots\right]$$

$$f''(A) = \frac{1}{(0.5)^2}\left[5.25 + 0.75 + 0\right]$$

$$f''(A) = \frac{1}{0.25}\left[6.00\right]$$

$$\boxed{f''(A) = 24}$$

2) From the following table, obtain the value of $\frac{d^2y}{dx^2}$ at the point $x = 0.96$ and $x = 1.04$. Using appropriate formula.

Soln:

Table:

x (x_0)	y (y_0)	Δy (Δy_0)	$\Delta^2 y$ ($\Delta^2 y_0$)	$\Delta^3 y$ ($\Delta^3 y_0$)	$\Delta^4 y$ ($\Delta^4 y_0$)
0.96	0.7825	−0.0086	−0.0002	0.0002	−0.0004
0.98	0.7739	−0.0088	0	−0.0002	
1.00	0.7651	−0.0088	−0.0002		
1.02	0.7563	−0.0090			
1.04	0.7473				
(x_0)	(y_0)	(∇y_0)	($\nabla^2 y_0$)	($\nabla^3 y_0$)	($\nabla^4 y_0$)

forward: $x_0 = 0.96$ $h = 0.98 - 0.96 = 0.02$

$$f'(0.96) = \frac{1}{0.02}\left[-0.0086 - \frac{1}{2}(-0.0002) + \frac{1}{3}(0.0002) - \frac{1}{4}(-0.0004)\right]$$

$=$

$$f''(0.96) = \frac{1}{(0.02)^2}\left[-0.0002 - 0.0002 + \frac{11}{12}(-0.0004)\right]$$

$$= \frac{1}{0.0004}\left[-0.0002 - 0.0002 - 0.000366\right]$$

$$f''(0.96) = -1.091666$$

Backward: $x_0 = 1.04$ $h = 0.02$.

H.W:

1) find $f'(0.01)$ and f' and $f''(0.06)$

x	0.01	0.02	0.03	0.04	0.05	0.06
$y=f(x)$	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

2) find f' & $f''(1.0)$ and f' and $f''(1.6)$

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
$y=f(x)$	7.989	8.403	8.781	9.129	9.451	9.750	10.03