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DEPARTMENT OF MATHEMATICS

21UCR304: BUSINESS CALCULUS AND FINANCIAL COMPUTATION

Life Annuities and Life Insurance

Ms.P.DEVIE ABIRAMI

Assistant Professor,

Department of Mathematics

Introduction to Life Annuities & Life Insurance

Life contingencies are financial contracts where payments depend on whether a person is alive or dead.

They combine two disciplines: Compound Interest (time value of money) + Probability/Statistics (mortality).

Life Annuity: Series of periodic payments made as long as the person (annuitant) is alive.

Life Insurance: A lump sum (death benefit) paid to beneficiaries upon the death of the insured.

The Actuary calculates the fair price (premium) of these contracts using mortality tables and interest theory.

Key notation: (x) denotes a life aged x; l_x = number of survivors at age x from an initial cohort.

Mortality Tables

- A mortality table (life table) is a statistical table showing the probability of death at each age.
- l_x: Number of lives alive at exact age x (out of initial l_0, often 100,000).
- d_x: Number of deaths between age x and x+1. $d_x = l_x - l_{x+1}$
- q_x: Probability of death within a year for age x. $q_x = d_x / l_x$
- p_x: Probability of surviving from age x to x+1. $p_x = 1 - q_x = l_{x+1}/l_x$

Age (x)	l_x	d_x	q_x
30	99000	150	0.0015
40	97000	300	0.0031
50	93000	700	0.0075
60	85000	1500	0.0176
70	70000	3000	0.0429

Pure Endowment

- A Pure Endowment is a contract that pays a sum S after n years, BUT ONLY IF the policyholder is alive at that time.
- If the person dies before n years, nothing is paid.
- Net Single Premium (NSP): The lump sum amount paid TODAY to receive S after n years if alive.
- $nE_x = v^n \cdot (l_{x+n} / l_x)$ — probability of surviving n years × present value factor.
- $l_{x+n} / l_x = {}_np_x$ = probability that (x) survives n more years.
- Example: Age 30, n=10 years, i=5%, S=₹1,00,000
- l_{30}=99,000; l_{40}=97,000 → NSP = 1,00,000 × $(1/1.05^{10})$ × (97,000/99,000)
- NSP = 1,00,000 × 0.6139 × 0.9798 ≈ ₹60,144

NET SINGLE PREMIUM

$$ {}_nE_x = v^n \times (l_{x+n}/l_x) $$
$$ = v^n \times {}_np_x $$

Life Annuities — Types & Concepts

Life Annuity: A series of payments made periodically as long as the annuitant is alive.

WHOLE LIFE ANNUITY-DUE ($\ddot{a}_x$): Payments made at the BEGINNING of each year for life.

WHOLE LIFE ANNUITY-IMMEDIATE (a_x): Payments made at the END of each year for life.

TEMPORARY LIFE ANNUITY ($a_{x:\overline{n}|}$): Payments for at most n years, provided annuitant is alive.

DEFERRED LIFE ANNUITY ($_{m|}a_x$): Payments begin after m years of deferral, continue for life.

Net Single Premium of whole life annuity (immediate):

$a_x = \Sigma\, v^k \cdot (l_{x+k} / l_x)$, summed for $k = 1$ to $\omega - x$ (ω = limiting age)

Higher interest rate → lower annuity value. Higher mortality → lower annuity value.

Life Annuity — Formulas

- Whole Life Annuity-Immediate: pays 1 at end of each year for life.
- $a_x = \Sigma\, v^k \cdot {}_kp_x$ (k = 1 to ω−x)
- Whole Life Annuity-Due: pays 1 at start of each year for life.
- $\ddot{a}_x = \Sigma\, v^k \cdot {}_kp_x$ (k = 0 to ω−x−1)
- Relationship: $\ddot{a}_x = 1 + a_x$ (since first payment immediate)
- n-year Temporary Annuity: $a_{x}:\bar{n}| = \Sigma\, v^k \cdot {}_kp_x$ (k = 1 to n)
- Deferred Annuity: ${}_m|a_x = a_x - a_x:\bar{m}|$
- Commutation Function: $D_x = v^x \cdot l_x$; $N_x = \Sigma\, D_{x+k}$ (k=0 to ∞)
- $a_x = (N_{x+1}) / D_x$ ← used with commutation tables for quick calculation.

COMMUTATION

$a_x = N_{x+1} / D_x$

$D_x = v^x \cdot l_x$

Life Insurance — Types

Whole Life Insurance

- Whole Life Insurance: Death benefit paid whenever death occurs.
- NSP (Whole Life) = $A_x = \Sigma\, v^{k+1} \cdot d_{x+k} / l_x$ ($k = 0$ to $\omega - x - 1$)
- Using commutation: $A_x = M_x / D_x$
- $M_x = \Sigma\, C_{x+k}$; $C_x = v^{x+1} \cdot d_x$
- Benefit: 1 (or sum insured) paid at end of year of death.

Term & Endowment Insurance

- n-Year Term Insurance: Death benefit ONLY if death occurs within n years.
- $A^1_{x:\bar{n}|} = (M_x - M_{x+n}) / D_x$
- n-Year Endowment Insurance: Death benefit OR survival benefit (whichever occurs first).
- $A_{x:\bar{n}|} = A^1_{x:\bar{n}|} + {}_nE_x = (M_x - M_{x+n} + D_{x+n}) / D_x$
- Deferred Whole Life: ${}_{n|}A_x = M_{x+n} / D_x$

Life Insurance — Commutation Formulas Summary

- Commutation Functions (used to simplify calculations):
- $D_x = v^x \cdot l_x$ (Discount × Survivors)
- $C_x = v^{x+1} \cdot d_x$ (Discount × Deaths)
- $M_x = \Sigma C_x$ (from x to ω) → $M_x = C_x + M_{x+1}$
- $N_x = \Sigma D_x$ (from x to ω) → $N_x = D_x + N_{x+1}$
- Whole Life Insurance: $A_x = M_x / D_x$
- Term Insurance: $A^1_{x}:\bar{n}| = (M_x - M_{x+n}) / D_x$
- Endowment Insurance: $A_x:\bar{n}| = (M_x - M_{x+n} + D_{x+n}) / D_x$

KEY FORMULAS

$$A_x = M_x / D_x$$
$$A^1_{x}:\bar{n}| = (M_x - M_{x+n})/D_x$$

Relationship: Life Annuity & Life Insurance

Key Identity:

$$A_x + d \cdot \ddot{a}_x = 1 \quad \text{where } d = i/(1+i) \text{ is the discount rate}$$

- This identity means: the cost of 1 unit whole life insurance + cost of annuity-due = 1.
- Derivation: Every person either dies at some year k, or survives. The two contracts together cover both outcomes completely.
- Rearranging: $\ddot{a}_x = (1 - A_x) / d$ ← compute annuity FROM insurance.
- Rearranging: $A_x = 1 - d \cdot \ddot{a}_x$ ← compute insurance FROM annuity.
- Similarly for temporary versions: $A^1_{x:\bar{n}|} + d \cdot \ddot{a}_{x:\bar{n}|} = 1 - {}_nE_x$
- These relationships are extremely useful in exam problems — memorize them!

Annual Premium Policies

- Instead of paying a single lump sum (NSP), the policyholder pays a level annual premium P.
- The premium is paid at the beginning of each year as long as the policy is in force.
- Equivalence Principle: PV of premiums = PV of benefits.
- For Whole Life Insurance (Sum Assured = S):
- $P \cdot \ddot{a}_x = S \cdot A_x \rightarrow P = S \cdot A_x / \ddot{a}_x$
- For n-Year Endowment Insurance:
- $P \cdot \ddot{a}_x{:}\bar{n}| = S \cdot A_x{:}\bar{n}| \rightarrow P = S \cdot A_x{:}\bar{n}| / \ddot{a}_x{:}\bar{n}|$
- For n-Year Term Insurance:
- $P \cdot \ddot{a}_x{:}\bar{n}| = S \cdot A^1{}_x{:}\bar{n}| \rightarrow P = S \cdot A^1{}_x{:}\bar{n}| / \ddot{a}_x{:}\bar{n}|$

ANNUAL PREMIUM

$P = S \cdot A_x / \ddot{a}_x$
(Whole Life)

$P = S \cdot A_x{:}\bar{n}| / \ddot{a}_x{:}\bar{n}|$
(Endowment)

Worked Example — Annual Premium

Problem: A man aged 35 takes a 20-year endowment policy with sum assured ₹1,00,000. Given: $i = 6\%$, $M_{35} = 480$, $M_{55} = 320$, $D_{35} = 2200$, $D_{55} = 900$, $N_{35} = 30000$, $N_{55} = 11000$. Find the annual premium.

- **Step 1 — Find $A_x{:}\bar{n}|$ (Endowment Insurance NSP for sum ₹1):**

$A_{35}{:}20| = (M_{35} - M_{55} + D_{55}) / D_{35} = (480 - 320 + 900) / 2200 = 1060 / 2200 \approx 0.4818$

- **Step 2 — Find $\ddot{a}_x{:}\bar{n}|$ (Annuity-due for 20 years):**

$\ddot{a}_{35}{:}20| = (N_{35} - N_{55}) / D_{35} = (30000 - 11000) / 2200 = 19000 / 2200 \approx 8.636$

- **Step 3 — Calculate Annual Premium P:**

$P = S \times A_{35}{:}20| / \ddot{a}_{35}{:}20| = 1{,}00{,}000 \times 0.4818 / 8.636 \approx$ ₹5,578 per year

- Interpretation: By paying ₹5,578 each year, the policyholder is covered for ₹1,00,000.

Summary of Commutation Functions

Symbol	Formula	Meaning	Used In
D_x	$v^x \cdot l_x$	Discounted survivors at age x	All policies
C_x	$v^{x+1} \cdot d_x$	Discounted deaths in year x	Insurance
N_x	ΣD_{x+k} (k=0→∞)	Sum of D_x values from x onwards	Annuities
M_x	ΣC_{x+k} (k=0→∞)	Sum of C_x values from x onwards	Insurance
A_x	M_x / D_x	NSP of Whole Life Insurance	Life Insurance
a_x	N_{x+1} / D_x	NSP of Whole Life Annuity	Life Annuity
$\ddot{a}_x$	N_x / D_x	NSP of Whole Life Annuity-due	Life Annuity

Unit 5.1 — Formula Quick Reference Card

CONTINGENT PAYMENTS

$E[X] = \Sigma x_i \cdot p_i$

$PV = p \times C \times v^n$

$v = 1/(1+i)^n$

$\Sigma p_i = 1$

MORTALITY

$p_x = l_{x+1} / l_x$

$q_x = d_x / l_x = 1 - p_x$

$d_x = l_x - l_{x+1}$

${}_np_x = l_{x+n} / l_x$

PREMIUMS

$P(WL) = S \cdot A_x / \ddot{a}_x$

$P(End) = S \cdot A_{x:\bar{n}|} / \ddot{a}_{x:\bar{n}|}$

$A_x + d \cdot \ddot{a}_x = 1$

$d = i/(1+i)$

THANK YOU