SNS COLLEGE OF TECHNOLOGY

An Autonomous Institution
Coimbatore-35

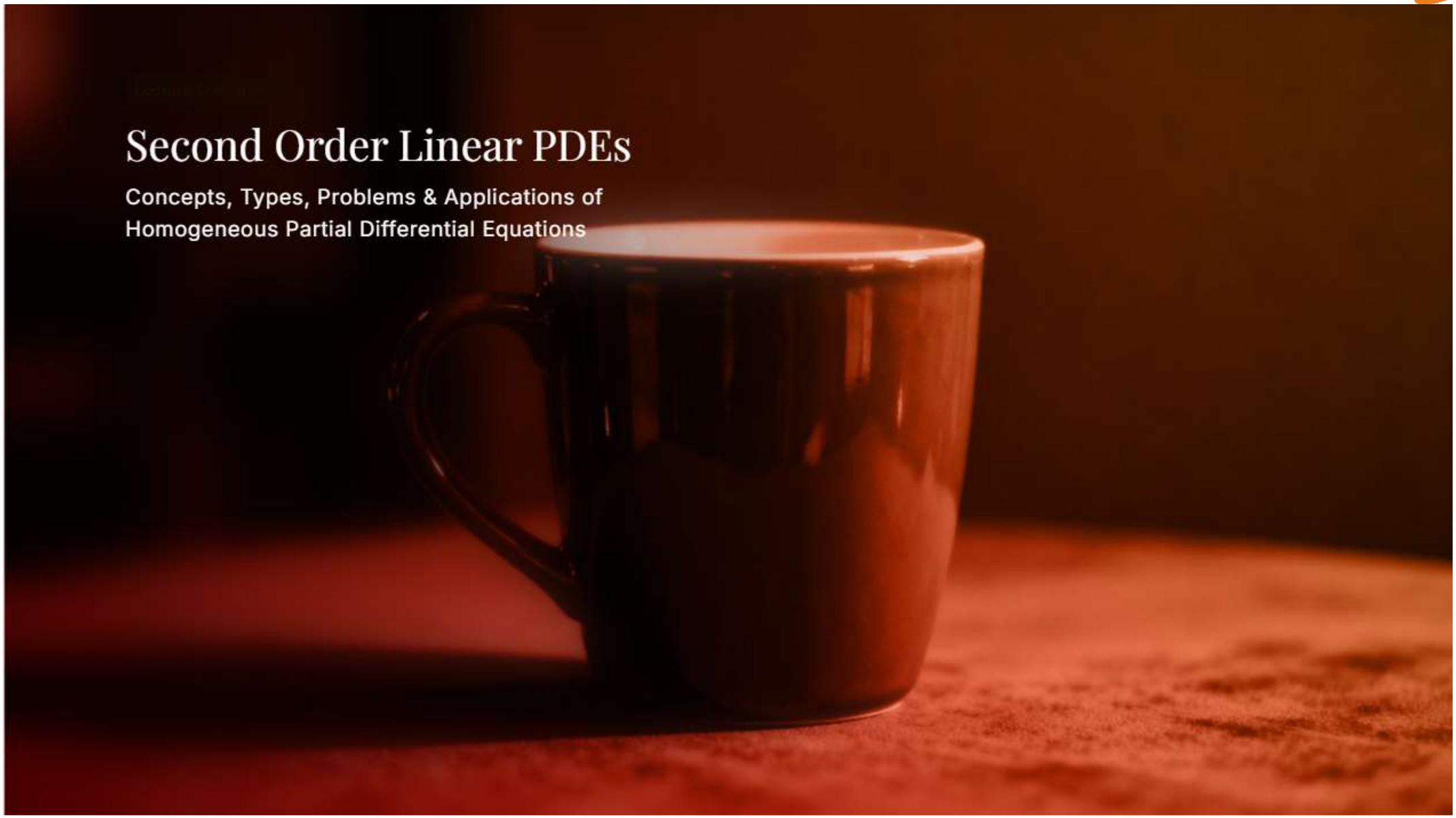
Department of Mathematics

23MAT103 –DIFFERENTIAL EQUATIONS AND TRANSFORMS

I B.E./ B.Tech. / II SEMESTER

UNIT III : Partial Differential Equations

Topic :Linear Partial differential equations of second order with constant coefficient


Second Order Linear PDEs
Concepts, Types, Problems & Applications of
Homogeneous Partial Differential Equations

Fundamental Concept Overview

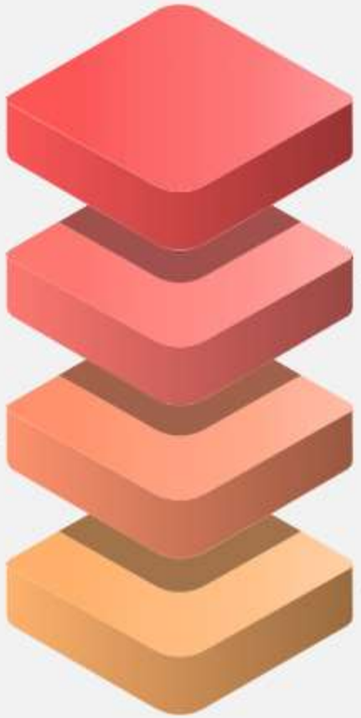
Understanding the core attributes of second-order linear operators

Order

Defined by the highest partial derivative present in the equation, specifically the second order for this study.

Constant Coefficients

The multipliers (A, B, C, etc.) of the partial derivatives are constants rather than functions of independent variables.



Linearity

The dependent variable and its derivatives appear only to the first power and are not multiplied together.

Homogeneous Nature

The equation equals zero ($f(x,y) = 0$), meaning every term contains the dependent variable or its derivatives.

General Form

The General Mathematical Form

Structural breakdown of the standard second-order linear PDE The standard form is expressed as: A u_xx + B u_xy + C u_yy + D u_x + E u_y + F u = 0. Here, A, B, and C represent the principal part coefficients, while D, E, and F represent lower-order terms. This structure is analogous to the general equation of a conic section in geometry.

Classification via Discriminant

Categorizing PDEs based on the discriminant $B^2 - 4AC$

Elliptic

Condition: $B^2 - 4AC < 0$. Characterized by steady-state processes and equilibrium configurations.

Parabolic

Condition: $B^2 - 4AC = 0$. Represents diffusion processes and time-dependent heat conduction.

Hyperbolic

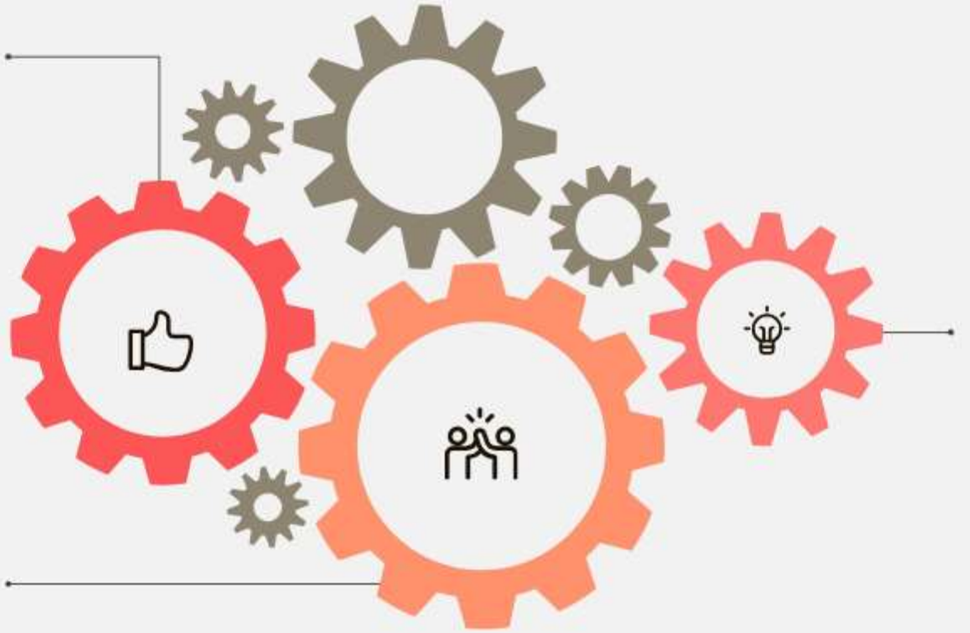
Condition: $B^2 - 4AC > 0$. Describes vibration, oscillation, and wave propagation phenomena.

Type 1: Elliptic Equations

Representing equilibrium and steady-state phenomena

Steady-State Heat

Describes temperature distributions that no longer change over time, known as the *Laplace Equation*.



Electrostatics

Models electric potential in a region free of charge, ensuring a smooth and continuous field.

Soap Films

Minimal surfaces where the tension is balanced at every point, visually represented by smooth membranes.

Type 2: Parabolic Equations

Modeling diffusion and heat conduction over time

Heat Diffusion

Governed by the *Heat Equation*, showing how thermal energy spreads through a medium over time.

Chemical Concentration

Models how a substance disperses through a fluid from high concentration to low concentration areas.

Irreversibility

Solutions exhibit a 'one-way' flow in time, where initial sharp differences are gradually smoothed out.

Type 3: Hyperbolic Equations

Visualizing wave propagation and oscillations

Wave Propagation

Describes how a disturbance travels through a medium, such as sound waves or ripples in water.

Vibrating Strings

Classic mechanical application where a string's displacement is governed by the *Wave Equation*.

Finite Speed

Unlike elliptic equations, information in hyperbolic systems travels at a specific, finite speed.

Solution Methodology

Step-by-step approach to solving homogeneous PDEs

Operator Substitution

Replace partial derivatives with operators: $D = \partial/\partial x$ and $D' = \partial/\partial y$ to form the symbolic equation.

Auxiliary Equation

Set $D = m$ and $D' = 1$ in the principal part to obtain the algebraic auxiliary equation $f(m, 1) = 0$.

Find Roots

Solve for the roots $m_1, m_2, \ldots$. Roots can be real and distinct, repeated, or complex conjugates.

General Solution

Form the Complementary Function (CF) using the roots: $z = \Sigma\, \varphi_i(y + m_i x)$.

Summary Table

Comparison of Equation Types

Quick reference for classification and physical examples

Type	Discriminant	Physical Model	Example Equation
Elliptic	$B^2 - 4AC < 0$	Steady State	$u_{xx} + u_{yy} = 0$
Parabolic	$B^2 - 4AC = 0$	Heat Diffusion	$u_t = \alpha u_{xx}$
Hyperbolic	$B^2 - 4AC > 0$	Wave Motion	$u_{tt} = c^2 u_{xx}$

Pre-assessment - Quiz

1. The general form of a second-order linear PDE with constant coefficients is:

A) $ax + by + c = 0$

B) $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$

C) $u_x + u_y = 0$

D) $x^2 + y^2 = 1$

✅ **Answer:** B

2. The PDE $u_{xx} + u_{yy} = 0$ is called:

A) Wave equation

B) Heat equation

C) Laplace equation

D) Diffusion equation

✅ **Answer:** C

References

- https://math.libretexts.org/Bookshelves/Differential_Equations/Introduction_to_Partial_Differential_Equations_%28Herman%29/02%3A_Second_Order_Partial_Differential_Equations?utm_source=chatgpt.com

- Classification of Second Order PDEhttps://math.libretexts.org/Bookshelves/Differential_Equations/Introduction_to_Partial_Differential_Equations_%28Herman%29/02%3A_Second_Order_Partial_Differential_Equations?utm_source=chatgpt.coms - Mathematics LibreTexts

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